TD 2 (SMA_S6)

Ex 1 1) $\forall k \in \{1,\dots,n\}$: $\begin{cases} P(X=k) = \frac{1}{n} \\ P(Y=k) = \frac{1}{n} \end{cases}$

$$P(X=Y) = P(\exists k \in [1:n] \,;\, X=k, Y=k)$$
$$= P\left(\bigcup_{k=1}^{n} (X=k, Y=k)\right)$$
$$= \sum_{k=1}^{n} P(X=k, Y=k)$$
$$= \sum_{k=1}^{n} P(X=k)\,P(Y=k) \quad \left(\text{X et Y sont indépendantes}\right)$$
$$= \sum_{k=1}^{n} \frac{1}{n} \times \frac{1}{n} = \frac{1}{n^2} \sum_{k=1}^{n} 1 = \frac{n}{n^2} = \boxed{\frac{1}{n}}$$

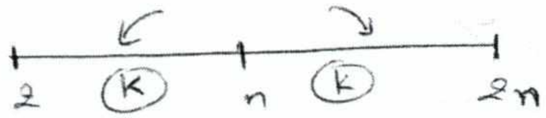
2) $$P(\Omega) = P((X \le Y) \cup (X \geq Y))$$
$$= \underbrace{P(X \le Y) + P(X \geq Y)}_{\text{la symétrie}} - P((X \le Y) \cap (X \geq Y))$$

$$\Rightarrow \quad 1 = 2P(X \geq Y) - P(X=Y)$$
$$\Rightarrow P(X \geq Y) = \frac{1 + P(X=Y)}{2} = \frac{1+\frac{1}{n}}{2}$$
$$\Rightarrow \boxed{P(X \geq Y) = \frac{n+1}{2n}}$$

3) X et Y prennent leurs valeurs dans $\{1,\dots,n\}$

$\Rightarrow Z = X+Y$ prend ses valeurs dans $\{2,\dots,2n\}$

cherchons $P(Z=k)$, $\forall k \in \{2,\dots,2n\}$??



$$P(Z=k) = P(X+Y=k) = P\left(\exists i \in \{1,\dots,k-1\}, X=i, Y=k-i\right)$$

$$= P\left(\bigcup_{i=1}^{k-1} (X=i, Y=k-i)\right)$$

$$= \sum_{i=1}^{k-1} P(X=i)\, P(Y=k-i)$$

$P(X=i) \neq 0 \Rightarrow 1 \leq i \leq n$

$P(Y=k-i) \neq 0 \Rightarrow 1 \leq k-i \leq n \Rightarrow k-n \leq i \leq k-1$

- 1er cas : $k \leq n \Rightarrow k-n \leq 0 \Rightarrow 1 \leq i \leq k-1$

$$\Rightarrow P(Z=k) = \sum_{i=1}^{k-1} \frac{1}{n} \times \frac{1}{n} = \frac{1}{n^2} \sum_{i=1}^{k-1} 1 = \frac{k-1}{n^2}$$

- 2ème cas : $k > n \Rightarrow \begin{cases} n \leq k-1 \\ k-n > 0 \end{cases} \Rightarrow k-n \leq i \leq n$

$$\Rightarrow P(Z=k) = \sum_{i=k-n}^{n} \frac{1}{n} \times \frac{1}{n} = \frac{2n-k+1}{n^2}$$

EX 2 1) a) $E(X) = \sum_{n=0}^{+\infty} n\, P(X=n)$

$$= \sum_{n=0}^{+\infty} n\, e^{-\lambda} \frac{\lambda^n}{n!} = \sum_{n=1}^{+\infty} \frac{e^{-\lambda} \lambda^n}{(n-1)!}$$

changement de variables : $n-1 \longrightarrow n'$

$$= \sum_{n'=0}^{+\infty} \frac{e^{-\lambda} \lambda^{n'+1}}{n'!} = \lambda e^{-\lambda} \sum_{n'=0}^{+\infty} \frac{\lambda^{n'}}{n'!} = \lambda e^{-\lambda} e^{\lambda} = \boxed{\lambda}$$

b) • $E(X(X-1)) = \sum_{n=0}^{+\infty} n(n-1)\, P(X=n)$

$$= \sum_{n=2}^{+\infty} n(n-1)\, e^{-\lambda} \frac{\lambda^n}{n!} = \sum_{n=2}^{+\infty} e^{-\lambda} \frac{\lambda^n}{(n-2)!}$$

changement de variables $n-2 \to n'$

$$= \sum_{n'=0}^{+\infty} e^{-\lambda} \frac{\lambda^{n'+2}}{n'!} = \lambda^2 e^{-\lambda} \sum_{n'=0}^{+\infty} \frac{\lambda^{n'}}{n'!} = \lambda^2 e^{-\lambda} . e^{\lambda} = \boxed{\lambda^2}$$

• $E(X(X-1)) = E(X^2) - E(X) = \lambda^2$

$$\Rightarrow E(X^2) = \lambda^2 + E(X) = \boxed{\lambda^2 + \lambda}$$

$$\Rightarrow V(X) = E(X^2) - (E(X))^2 = \lambda^2 + \lambda - \lambda^2 = \boxed{\lambda}$$

c) (+) $E\left(\frac{X!}{(X-k)!} \mathbb{1}_{(X \geq k)} \right) = \sum_{n=0}^{+\infty} \frac{n!}{(n-k)!} \mathbb{1}_{(n \geq k)} P(X=n)$

$$= \sum_{n=k}^{+\infty} \frac{\cancel{n!}}{(n-k)!} e^{-\lambda} \frac{\lambda^n}{\cancel{n!}}$$

$\boxed{n \geq k \Leftrightarrow \mathbb{1}_{(n \geq k)} = 1}$

$n' = n-k$

$$= e^{-\lambda} \sum_{n'=0}^{+\infty} \frac{\lambda^{n'+k}}{n'!} = e^{-\lambda} \lambda^k e^{\lambda} = \boxed{\lambda^k}$$

2) a) $Y \geq 0 \Rightarrow E(Y)$ est bien définie.

$$E(Y) = E(2^{-X}) = \sum_{n=0}^{+\infty} 2^{-n} e^{-\lambda} \frac{\lambda^n}{n!} = e^{-\lambda} \sum_{n=0}^{+\infty} \frac{\left(\frac{\lambda}{2}\right)^n}{n!}$$

$$= e^{-\lambda} . e^{\lambda/2} = \boxed{e^{-\lambda/2}} < +\infty .$$

$E(Y) < +\infty \Rightarrow Y$ est intégrable $(Y \in L^1(\Omega))$.

<u>Rappel :</u> $e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$

b) X prend ses valeurs dans $\mathbb{N} \Rightarrow X(\Omega) = \mathbb{N}$.

$\Rightarrow Y$ est à valeurs dans l'ensemble dénombrable :

$$Y(\Omega) = \{2^{-n} ;\ n \in \mathbb{N}\}$$

$$\Rightarrow P_Y = \sum_{n=0}^{+\infty} P(Y = 2^{-n})\, \delta_{2^{-n}} \text{ avec :}$$

$$P(Y = 2^{-n}) = P(2^{-X} = 2^{-n}) = P(X = n) = \frac{e^{-\lambda} \lambda^n}{n!}$$

$$\Rightarrow \boxed{P_Y = \sum_{n=0}^{+\infty} e^{-\lambda} \frac{\lambda^n}{n!} \delta_{2^{-n}}}$$

c) $E|Y'| = E|(-2)^{-X}| = E|2^{-X}| = E(Y) < +\infty$

$\Rightarrow Y'$ intégrable $(Y' \in L^1(\Omega))$.

et $E(Y') = E((-2)^{-X}) = \sum_{n=0}^{+\infty} (-2)^{-n} P(X = n)$

$$= \sum_{n=0}^{+\infty} (-2)^{-n} e^{-\lambda} \frac{\lambda^n}{n!}$$

3) Z prend ses valeurs dans $\mathbb{Z}$: $(Z(\Omega) = \mathbb{Z})$

Calculons $P(Z = p)$, $\forall p \in \mathbb{Z}$??

- $P(Z = p) = P(Z = p,\ X \text{ pair}) + P(Z = p,\ X \text{ impair})$

$= P\left(\frac{X}{2} = p,\ X \text{ pair}\right) + P\left(\frac{1-X}{2} = p,\ X \text{ impair}\right)$

$= P(X = 2p) + P(X = 1 - 2p)$

car $(2p)$ et $(1-2p)$ sont pairs.

- $X \sim \mathcal{P}(\lambda) \Rightarrow P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$, $\forall k \in \mathbb{N}$ $(k \geq 0)$.

donc : $P(X=2p) = \begin{cases} 0 & ; \ p < 0 \\ e^{-\lambda} \dfrac{\lambda^{2p}}{(2p)!} & ; \ p \geq 0 \end{cases}$

et $P(X=1-2p) = \begin{cases} 0 & ; \ 1-2p<0 \Rightarrow p \geq 1 \ \left(\begin{smallmatrix}\text{car} \\ p \in \mathbb{Z}\end{smallmatrix}\right) \\ e^{-\lambda} \dfrac{\lambda^{1-2p}}{(1-2p)!} & , \ \text{sinon} \ (p \leq 0) \ (p<1) \end{cases}$

finalement :

$$P(Z=p) = \begin{cases} e^{-\lambda} \dfrac{\lambda^{1-2p}}{(1-2p)!} & ; \ (p<0) \\ \dfrac{e^{-\lambda}\lambda^{1-2p}}{(1-2p)!} + \dfrac{e^{-\lambda}\lambda^{2p}}{(2p)!} = e^{-\lambda}(\lambda+1) & ; \ (p=0) \\ e^{-\lambda} \dfrac{\lambda^{2p}}{(2p)!} & ; \ p>0 \end{cases}$$

4) a) $X(\Omega) = \mathbb{N} \Rightarrow U(\Omega) = \mathbb{Z}$

et $U = 4[X/2] - 2X + 1 = \begin{cases} 4\dfrac{X}{2} - 2X + 1 & ; \ X \text{ pair} \\ 4\dfrac{X-1}{2} - 2X + 1 & ; \ X \text{ impair} \end{cases}$

$= \begin{cases} 1 & ; \ X \text{ pair} \\ -1 & ; \ X \text{ impair} \end{cases}$

$\Rightarrow \boxed{U(\Omega) = \{-1, 1\}}$

et $\boxed{P_U = P(U=1)\,\delta_1 + P(U=-1)\,\delta_{-1}}$

$P(U=-1) = P(X \text{ impair}) = e^{-\lambda} \sum_{p=0}^{+\infty} \dfrac{\lambda^{2p+1}}{(2p+1)!}$

$= e^{-\lambda} \sinh(\lambda) = e^{-\lambda}\left(\dfrac{e^{\lambda} - e^{-\lambda}}{2}\right) = \boxed{\dfrac{1-e^{-2\lambda}}{2}}$

et $P(U=1) = 1 - P(U=-1) = 1 - \left(\dfrac{1-e^{-2\lambda}}{2}\right)$

$= \boxed{\dfrac{1+e^{-2\lambda}}{2}}$

Ex 3 1 • $X \sim \mathcal{B}(n,p)$ (loi Binomiale)

$$\Rightarrow \begin{cases} X(\Omega) = \{0, 1, \dots, n\} \\ P(X=k) = C_n^k p^k (1-p)^{n-k} \ ; \ \forall k \in X(\Omega) \end{cases}$$

$$g_X(s) = E(s^X) = \sum_{k=0}^{n} s^k P(X=k)$$

$$= \sum_{k=0}^{n} C_n^k (ps)^k (1-p)^{n-k} = \boxed{(sp+1-p)^n}$$

d'après formule de Binôme $(a+b)^n = \sum_{k=0}^{n} C_n^k a^k b^{n-k}$

$$g_X(s) = (sp+1-p)^n$$

$$g'_X(s) = np(sp+1-p)^{n-1} \Rightarrow E(X) = g'_X(1) = \boxed{np}$$

$$g''_X(s) = n(n-1)p^2(sp+1-p)^{n-2} \Rightarrow g''_X(1) = \boxed{n(n-1)p^2}$$

$$\Rightarrow V(X) = g''_X(1) + g'_X(1) - (g'_X(1))^2$$
$$= n(n-1)p^2 + np - n^2p^2 = \boxed{np(1-p)}$$

• $X \sim \mathcal{B}(p)$ (loi Bernoulli $\Rightarrow \boxed{n=1}$

$$\begin{cases} X(\Omega) = \{0, 1\} \\ P(X=1) = p \text{ et } P(X=0) = 1-p \end{cases}$$

$$\Rightarrow \begin{cases} E(X) = 1 \times p = \boxed{p} \\ V(X) = 1 \times p(1-p) = \boxed{p(1-p)} \end{cases}$$ $n=1$

2 $X \sim Geo(p)$ (géométrique)

$$\Rightarrow \begin{cases} X(\Omega) = \mathbb{N}^* = \{1, 2, \dots\} \\ P(X=k) = p(1-p)^{k-1} \ , \ \forall k \in \mathbb{N}^* \end{cases}$$

$$g_X(s) = E(s^X) = \sum_{k=1}^{+\infty} s^k p(X=k)$$

$$= \sum_{k=1}^{+\infty} s^k p(1-p)^{k-1}$$

$$k' = k-1 \Rightarrow g_X(s) = \sum_{k'=0}^{+\infty} s^{k'+1} p(1-p)^{k'}$$

$$= sp \sum_{k'=0}^{+\infty} (s(1-p))^{k'} \quad \text{(série géométrique)}$$

$$\boxed{g_X(s) = \frac{sp}{1-s(1-p)}}$$

$$g_X'(s) = \frac{p(1-s(1-p)) + sp(1-p)}{(1-s(1-p))^2} \Rightarrow g_X'(1) = \frac{p}{p^2} = \frac{1}{p}$$

$$\Rightarrow E(X) = g_X'(1) = \frac{1}{p}$$

$$g_X'(s) = \frac{p}{(1-s(1-p))^2} \Rightarrow g_X''(s) = \frac{2p(1-p)(1-s(1-p))}{(1-s(1-p))^4}$$

$$= \frac{2p(1-p)}{(1-s(1-p))^3}$$

$$\Rightarrow g_X''(1) = \frac{2p(1-p)}{p^3} = \boxed{\frac{2(1-p)}{p^2}}$$

$$\Rightarrow V(X) = g_X''(1) + g_X'(1) - (g_X'(1))^2$$

$$= \frac{2(1-p)}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{2}{p^2} - \frac{2}{p} + \frac{1}{p} - \frac{1}{p^2}$$

$$\Rightarrow \boxed{V(X) = \frac{1-p}{p^2}}$$

③ $X \sim \mathcal{P}(\lambda) \Rightarrow \begin{cases} X(\Omega) = \mathbb{N} = \{0, 1, \dots\} \\ P(X=k) = e^{-\lambda} \dfrac{\lambda^k}{k!} \end{cases}$

$$\Rightarrow g_X(s) = E(s^X) = \sum_{k=0}^{+\infty} e^{-\lambda} \frac{\lambda^k}{k!} s^k$$

$$= e^{-\lambda} \sum_{k=0}^{+\infty} \frac{(s\lambda)^k}{k!} = e^{-\lambda} e^{s\lambda} = \boxed{e^{\lambda(s-1)}}$$

$$e^x = \sum_{k=0}^{+\infty} \frac{x^n}{n!}$$

$$\Rightarrow g'_X(1) = \lambda e^{\lambda(s-1)} \Rightarrow \boxed{g'_X(1) = E(X) = 1}$$

$$g''_X(1) = \lambda^2 e^{\lambda(s-1)} \Rightarrow g''_X(1) = \boxed{\lambda^2}$$

$$\Rightarrow V(X) = g''_X(1) + g'_X(1) - (g'_X(1))^2$$

$$= \lambda^2 + \lambda - \lambda^2 = \boxed{\lambda}$$

Ex④ ① $X \sim \mathcal{B}(p) \Rightarrow \begin{cases} X(\Omega) = \{0, 1\} \\ P(X=1) = p \\ P(X=0) = 1-p \end{cases}$

$Y \sim \mathcal{P}(1) \Rightarrow \begin{cases} Y(\Omega) = \mathbb{N} \\ P(Y=k) = \dfrac{e^{-1}}{k!} \end{cases}$ $(\lambda = 1)$

$\Rightarrow Z = XY$ prend ses valeurs dans $\mathbb{N}$

cherchons $P(Z=k)$, $\forall k \in \mathbb{N}$.

on distingue deux cas :

- $k \neq 0$ $\quad P(Z=k) = P(XY=k)$

$$= P(X=1, Y=k) \quad \left(\begin{array}{c} X \text{ et } Y \\ \text{indépendantes} \end{array}\right)$$

$$= P(X=1)\,P(Y=k)$$

$$= \boxed{p\,\frac{e^{-1}}{k!}}$$

- $k = 0$: $\quad P(Z=0) = 1 - P(Z \neq 0)$

$$= 1 - P(\exists k \neq 0, Z = k)$$

$$= 1 - P\left(\bigcup_{k=1}^{+\infty} (Z=k)\right)$$

$$= 1 - \sum_{k=1}^{+\infty} P(Z=k) \quad (k \neq 0)$$

$$= 1 - \sum_{k=1}^{+\infty} \frac{p e^{-1}}{k!} = 1 - \sum_{k=0}^{+\infty} \frac{p e^{-1}}{k!} + p e^{-1}$$

$$= 1 - p e^{-1} \left(\sum_{k=0}^{+\infty} \frac{1}{k!}\right) + p e^{-1}$$

$$= 1 - p e^{-1} \times e + p e^{-1} = \boxed{1 - p + p e^{-1}}$$

Car : $\boxed{e^x = \sum_{k=0}^{+\infty} \frac{x^n}{K!} \Rightarrow e = \sum_{k=0}^{+\infty} \frac{1}{K!}}$

$$\Rightarrow \boxed{P_Z = \sum_{k=1}^{+\infty} \frac{p e^{-1}}{k!} \delta_k + (1 - p + p e^{-1})\,\delta_0}$$

2 $\cdot$ $E(Z) = \sum_{k=0}^{+\infty} k\,P(Z=k)$

$= 0 \times P(Z=0) + \sum_{k=1}^{+\infty} k\,P(Z=k)$

$= \sum_{k=1}^{+\infty} k \frac{pe^{-1}}{k!} = pe^{-1} \sum_{k=1}^{+\infty} \frac{1}{(k-1)!}$

$k' = k-1 \Rightarrow = pe^{-1} \sum_{k=0}^{+\infty} \frac{1}{k!} = pe^{-1} \cdot e = \boxed{p}$

$\bullet\ V(Z) = E(Z^2) - (E(Z))^2$.

$\cdot\ E(Z^2) = \sum_{k=0}^{+\infty} k^2 P(Z=k) = \sum_{k=1}^{+\infty} k^2 P(Z=k)$ $(\Rightarrow k \neq 0)$

$= \sum_{k=1}^{+\infty} k^2 \frac{pe^{-1}}{k!} = pe^{-1} \sum_{k=1}^{+\infty} \frac{k}{(k-1)!}$

$\begin{cases} k' = k-1 \\ k = k'+1 \end{cases} \Rightarrow = pe^{-1} \sum_{k'=0}^{+\infty} \frac{(k'+1)}{k'!}$

$= pe^{-1} \left(\sum_{k'=0}^{+\infty} \frac{k'}{k'!} + \frac{1}{k'!} \right)$

$= pe^{-1} \left(\sum_{k'=1}^{+\infty} \frac{1}{(k'-1)!} + \sum_{k'=0}^{+\infty} \frac{1}{k'!} \right)$

$= pe^{-1} \left[\left(\sum_{k''=0}^{+\infty} \frac{1}{k''!} \right) + \sum_{k'=0} \frac{1}{k'!} \right]$

$k'' = k'-1$

$= pe^{-1}(e+e) = \boxed{2p}$

$\Rightarrow V(Z) = 2p - p^2 = \boxed{p(2-p)}$ $\Rightarrow$ $0 < p < 1 < 2$, $2-p > 0$

③ $g_Z(s) = E(s^Z) = \sum_{k=0}^{+\infty} s^k P(Z=k)$

$= s^0 P(Z=0) + \sum_{k=1}^{+\infty} s^k P(Z=k)$

$= (1-p+pe^{-1}) + \sum_{k=1}^{+\infty} s^k \frac{pe^{-1}}{k!}$

$= 1-p+pe^{-1} + pe^{-1}\left(\sum_{k=0}^{+\infty} \frac{s^k}{k!} - 1\right)$

$= 1-p+pe^{-1} + pe^{-1}e^{s} = \boxed{1-p+pe^{-1}(1+e^{s})}$

④ $g'_Z(s) = pe^{-1}e^{s} \Rightarrow g'_Z(1) = pe^{-1}e^{1} = \boxed{p}$

$\Rightarrow E(Z) = g'_Z(1) = \boxed{p}$

$g''_Z(s) = pe^{-1}e^{s} \Rightarrow g''_Z(1) = pe^{-1}e^{1} = \boxed{p}$

$\Rightarrow V(Z) = g''_Z(1) + g'_Z(1) - (g'_Z(1))^2$

$= p + p - p^2 = 2p - p^2 = \boxed{p(2-p)}$

EX⑤ : ① Méthode ①

$X \sim \mathcal{P}(\lambda) \Rightarrow \begin{cases} X(\Omega) = \mathbb{N} \\ \forall k;\ P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!} \\ g_X(s) = e^{\lambda(s-1)} \end{cases}$

$Y \sim \mathcal{P}(\beta) \Rightarrow \begin{cases} Y(\Omega) = \mathbb{N} \\ \forall k;\ P(Y=k) = e^{-\beta}\frac{\beta^k}{k!} \\ g_Y(s) = e^{\beta(s-1)} \end{cases}$

X et Y indépt $\Rightarrow g_{X+Y}(s) = g_X(s)\, g_Y(s)$

$$= e^{(\lambda+\beta)(s-1)}$$

$$\Rightarrow \boxed{Z = X+Y \sim \mathcal{P}(\lambda+\beta)}$$

Méthode 2 : $X(\Omega) = Y(\Omega) = Z(\Omega) = \mathbb{N}$

Calculer $P(Z=k)$, $\forall k \in \mathbb{N}$??

$$P(Z=k) = P(X+Y=k) = P(\exists i \ (X=i, Y=k-i))$$

$$= P\Big(\bigcup_i (X=i, Y=k-i)\Big)$$

$$= \sum_i P(X=i)\, P(Y=k-i)$$

$P(X=i) \neq 0 \Rightarrow i \in \mathbb{N} \Rightarrow i \geq 0$

$P(Y=k-i) \neq 0 \Rightarrow k-i \in \mathbb{N} \Rightarrow i \leq k$

$\Rightarrow \boxed{0 \leq i \leq k}$

$$\Rightarrow P(Z=k) = \sum_{i=0}^{k} P(X=i)\, P(Y=k-i)$$

$$= \sum_{i=0}^{k} e^{-\lambda} \frac{\lambda^i}{i!}\, e^{-\beta} \frac{\beta^{k-i}}{(k-i)!}$$

$$= \left(\sum_{i=0}^{k} C_k^i \lambda^i \beta^{k-i}\right) \frac{e^{-(\lambda+\beta)}}{k!} \qquad \boxed{C_k^i = \frac{k!}{i!\,(k-i)!}}$$

$$= \frac{e^{-(\lambda+\beta)}}{k!} (\lambda+\beta)^k \Rightarrow Z \sim \mathcal{P}(\lambda+\beta).$$

② $Z \sim \mathcal{P}(\lambda+\beta)$

$\Rightarrow$ cours : $\boxed{E(Z) = V(Z) = \lambda+\beta}$

EX ⑥ $X \sim Geo(p) \Rightarrow \begin{cases} X(\Omega) = \mathbb{N}^* \\ P(X=k) = p(1-p)^{k-1} \end{cases}$

$Y \sim Geo(q) \Rightarrow \begin{cases} Y(\Omega) = \mathbb{N}^* \\ P(Y=k) = q(1-p)^{k-1} \end{cases}$

$Z = \min(X,Y) \Rightarrow Z(\Omega) = \mathbb{N}^*$

① $P(X \geq k) = 1 - P(X < k) = 1 - P(X \leq k-1)$

$= 1 - \sum_{i=1}^{k-1} P(X=i) = 1 - \sum_{i=1}^{k-1} p(1-p)^{i-1}$

$i' = i-1$

$= 1 - p\sum_{i'=0}^{k-2} (1-p)^{i'}$

$= 1 - p\,\frac{1-(1-p)^{k-1}}{1-(1-p)}$

(Série géométrique)

$u_1 + \dots + u_p = u_1\left(\frac{1-q^{n-p+1}}{1-q}\right)$

$\Rightarrow \boxed{P(X \geq k) = (1-p)^{k-1}}$

② $P(Z \geq k) = P(\min(X,Y) \geq k)$

$= P(X \geq k, Y \geq k)$

$= P(X \geq k) \times P(Y \geq k)$

$= (1-p)^{k-1}(1-q)^{k-1}$

$\boxed{P(Z \geq k) = \left[(1-p)(1-q)\right]^{k-1}}$

- $F_Z(k) = P(Z \leq k)$

$$= 1 - P(Z > k) \qquad k \longleftrightarrow k+1$$
$$= 1 - P(Z \geqslant k+1)$$
$$= 1 - \left((1-p)(1-q)\right)^{k+1-1}$$

$$\Rightarrow \boxed{F_Z(k) = 1 - \left((1-p)(1-q)\right)^{k}}$$

- D'après 1 si $X \sim Geo(p)$

$$\Rightarrow \boxed{P(X \geqslant k) = (1-p)^{k-1}}$$

et on a $P(Z \geqslant k) = \left((1-p)(1-q)\right)^{k-1}$

$$= (1-p-q+pq)^{k-1}$$
$$\boxed{= \left[1 - (p+q-pq)\right]^{k-1}}$$

par identification : $\boxed{Z \sim Geo(p+q-pq)}$

4 $P(X<Y) = P(\exists i \in \mathbb{N}^*, Y=i, X<i)$

$$= P\left(\bigcup_{i=1}^{+\infty} (Y=i, X<i)\right)$$
$$= \sum_{i=1}^{+\infty} P(X<i)\, P(Y=i)$$
$$= \sum_{i=1}^{+\infty} \left[1 - P(X \geqslant i)\right] P(Y=i)$$
$$= \sum_{i=1}^{+\infty} P(Y=i) - \underbrace{\sum_{i=1}^{+\infty} (1-p)^{i-1}}_{\text{d'après } 1} q(1-q)^{i-1}$$

$$\begin{cases} Y \sim Geo(q) \\ P(Y=i) = q(1-q)^{i-1} \end{cases} \Rightarrow \boxed{\sum_{i=1}^{+\infty} P(Y=i) = 1}$$

$$\text{et } \sum_{i=1}^{+\infty} (1-p)^{i-1} q (1-q)^{i-1} = q \sum_{i=1}^{+\infty} \left[(1-p)(1-q)\right]^{i-1}$$

$$i' = i-1 \Rightarrow \; = q \sum_{i'=0}^{+\infty} \left[(1-p)(1-q)\right]^{i'}$$

$$= q \times \frac{1}{(1-p)(1-q)}$$

$$\sum_{i=0}^{+\infty} x^i = \frac{1}{1-x}$$

$$|x| < 1$$

$$\Rightarrow P(X<Y) = 1 - \frac{q}{(1-p)(1-q)}$$

$$\Rightarrow \boxed{P(X<Y) = \frac{1-p-q+pq}{(1-p)(1-q)}}$$

FIN