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UNIVERSITÉ MOULAY ISMAÏL

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FACULTÉ DES SCIENCES

Course notes

Advanced Biomedical Signal and Image Processing

Master: Plasturgy & Biomedical Engineering

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Summary

General introduction

The Advanced Biomedical Signal and Image Processing module, a key component of your Master's degree program at the UMI Science Faculty, Meknes. This course aims to equip you with the essential knowledge and skills required to navigate the rapidly evolving fields of biomedical signal and image processing. As technology continues to advance, analyzing and interpreting complex biomedical data becomes increasingly critical for healthcare professionals and researchers alike. The module is structured into three sections:

Section 1: Introduction to Digital Signal and Image Processing in which, we investigate the fundamental concepts of digital signal and image processing. You will learn about various types of signals, particularly biomedical signals, and explore techniques such as Fourier analysis, which is vital for understanding frequency components in signals. We will also cover image filtering, enhancement, and restoration, equipping you with tools to improve image quality. Additionally, you will gain insights into use detection, segmentation methods, wavelet transforms, and other advanced processing techniques, including clustering and classification, which are essential for effective data analysis.

Section 2: Processing of Biomedical Signals in which we aim on the specific processing techniques used for various biomedical signals. You will explore the electrical activities of cells and how these signals appear in various biomedical contexts, including electrocardiograms (ECGs), electroencephalograms (EEGs), and electromyograms (EMGs).Understanding these signals is crucial for diagnosing and monitoring various health conditions. During this section, you will learn with practical knowledge and analytical skills.

Section 3: Processing of Biomedical Images, this final part of the module addresses the principles and techniques involved in biomedical imaging. You will explore various imaging modalities, including computed tomography (CT), X-ray imaging, magnetic resonance imaging (MRI), ultrasound imaging, and Nuclear medicine. Each technique offers unique insights into biological processes and anatomical structures, and mastering these methods will enhance your ability to interpret and utilize imaging data effectively.

Through this module, you will develop a robust foundation in both theoretical and practical aspects of biomedical signal and image processing. By the end of the course, you will be well prepared to apply these skills in research or clinical settings, contributing to advancements in medical technology and patient care.

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Section 1: Introduction to Digital Signal and Image Processing

In this first part of the module, we will establish a solid foundation in digital signal and image processing, particularly focusing on their applications in the biomedical field. You will begin by exploring the concept of signals and the complexities of biomedical signal processing, which form the backbone of interpreting physiological data.

Next, we will delve into Fourier analysis, a powerful mathematical tool that enables you to analyze the frequency components of signals, revealing their behavior and characteristics. Building on this foundation, we will examine various techniques for image filtering, enhancement, and restoration, essential for improving image quality and facilitating accurate analysis.

Additionally, you will learn about edge detection and segmentation methods that help isolate and identify important features within images, critical steps in many biomedical imaging applications. We will also cover wavelet transforms, which provide a versatile approach to signal and image analysis, allowing for multi-resolution representation.

Finally, we will explore advanced signal and image processing methods, including clustering and classification techniques. These methods enable you to organize and interpret data effectively, enhancing your ability to draw meaningful conclusions from complex biomedical datasets. By the end of this section, you will gain a comprehensive understanding of the essential concepts and techniques that underpin digital signal and image processing in the biomedical context.

Chapter 1: Introduction to signals and systems

A. Signal

1. Definitions

A signal is a function that conveys information about a physical phenomenon. We represent a signal as a mathematical function of one or more independent variables, such as time (for temporal signals) or space (for spatial signals). Such as: electrical signals, sound waves, biomedical signals (Electroencephalography (EEG) and electrocardiography (ECG)(and digital data.

A system is a process or device that takes one or more signals as input, processes them, and produces one or more output signals. We classify systems as linear or nonlinear, time-invariant or time-variant, and continuous-time or discrete-time. Such as: amplifiers, filters, communication systems, and biomedical devices.

2. Digital signal processing in biomedical engineering

2.1. Definition:

Digital Signal Processing (DSP) in biomedical engineering involves the analysis, transformation, and interpretation of biomedical signals using digital techniques. These signals, such as ECG (Electrocardiogram), EEG (Electroencephalogram), and EMG (Electromyogram), we collect those signals from the human body and proceed for medical diagnosis, monitoring, and research. Digital signal processing plays a crucial role in improving the accuracy and efficiency of biomedical diagnostics and patient monitoring.

2.2. Key aspects of biomedical DSP:

Signal acquisition & preprocessing: Converting analog biomedical signals into digital form using Analog-to-Digital Conversion (ADC) and removing noise or artifacts.

Filtering & enhancement: Using digital filters (low-pass, high-pass, band-pass, notch filters) to eliminate unwanted noise and extract relevant signal components.

Feature extraction: Identifying key characteristics from biomedical signals, such as heart rate variability in ECG or brain wave patterns in EEG.

Signal classification & interpretation: Using machine learning, AI, and pattern recognition to classify biomedical conditions based on processed signals.

Applications in medical devices: Implementation in wearable health monitors, diagnostic equipment, and telemedicine solutions.

3. Communication and control systems

Communication systems involve the transmission, reception, and processing of information, such as:

Wireless communication: Mobile networks (4G, 5G), Wi-Fi, and Bluetooth.

Satellite communication: GPS, weather forecasting, and global broadcasting.

Optical fiber communication: High-speed internet using fiber-optic cables.

Biomedical communication: Wireless transmission of patient health data from wearable devices.

Control systems used to regulate and manage processes in various applications such as:

Automatic temperature control: HVAC systems regulating room temperature.

Industrial automation: Robotics and conveyor belt systems in manufacturing.

Biomedical control systems: Pacemakers regulating heartbeats and insulin pumps managing blood sugar levels.

Autonomous vehicles: Self-driving cars using sensors and control algorithms.

Both communication and control systems are essential in modern technology, enabling automation, efficiency, and connectivity across various industries.

4. Classification of signals

We classify signals based on different criteria. Understanding these classifications helps in analyzing and processing signals effectively in various applications, especially in biomedical engineering

4.1. Continuity:

Continuous-time signals: Defined for all time values (e.g., ECG, EEG signals).

Discrete-time signals: Defined only at specific time intervals (e.g., digital audio signals).

4.2. Determinism:

Deterministic signals: Completely defined by a mathematical function (e.g., sine wave).

Random (stochastic) signals: Have unpredictable variations (e.g., noise in biomedical signals).

4.3. Periodicity:

Periodic signals: Repeat over a fixed interval (e.g., heart rate waveform).

Aperiodic signals: Do not repeat in a fixed pattern (e.g., speech signals).

4.4. Symmetry:

Even Signals: Symmetric about the y-axis (e.g., cosine wave).

Odd Signals: Symmetric about the origin (e.g., sine wave).

4.5. Energy and power:

Energy Signals: Have finite energy but zero power (e.g., pulses).

Power Signals: Have finite power but infinite energy (e.g., sine waves).

4.6. Analog or digital nature:

Analog Signals: Continuous in amplitude and time (e.g., biological signals).

Digital Signals: Discrete in amplitude and time (e.g., digital ECG data).

5. Operations on signals

Various mathematical operations can be performed on signals to modify or analyze them. These operations are essential in signal processing, particularly in biomedical applications like ECG, EEG, and EMG analysis.

5.1. Basic Arithmetic Operations:

Addition: $y(t) = x_1(t) + x_2(t)$ (Combining two signals).

Subtraction: $y(t) = x_1(t) - x_2(t)$ (Finding the difference between signals).

Multiplication: $y(t) = x_1(t) * x_2(t)$ (Modulation in communication systems).

Scaling: $y(t) = Ax(t)$ (Amplification or attenuation of a signal).

5.2. Time-Domain Operations

Time Shifting: $y(t) = x(t - T)$ (Delaying or advancing a signal in time).

Time Scaling: $y(t) = x(at)$ (Compressing or expanding a signal in time).

- If $a > 1$, the signal is compressed.
- If $0 < a < 1$, the signal is stretched.

Time Reversal (Mirroring): $y(t) = x(-t)$ (Flipping a signal around the vertical axis).

5.3. Convolution (Continuous and Discrete)

Used to analyze system response and filter signals.

Continuous-Time Convolution: $y(t) = \int_{-\infty}^{\infty} x_1(\tau)h(t-\tau)d\tau$

Discrete-Time Convolution $y[n] = \sum_{k=-\infty}^{\infty} x_1[k]h[n-k]$

5.4. Correlation (Signal similarity)

Measures similarity between two signals over time.

Used in pattern recognition and biomedical signal analysis (e.g., comparing ECG patterns).

5.5. Differentiation and integration:

Differentiation: Enhances high-frequency components (e.g., detecting sharp changes in ECG).

Integration: Smoothens a signal (useful in noise reduction).

These operations are fundamental in signal processing, allowing for effective analysis, filtering, and interpretation in various engineering and biomedical applications.

6. Elementary signals: exponential, sinusoidal, and others

Elementary signals serve as the building blocks in signal processing and system analysis. Some of the most fundamental signals include:

6.1. Unit impulse (Dirac delta) signal δ(t)

Defined as an infinitely short and high pulse with an area of 1.

Used in system analysis as an input to determine system response.

Mathematically: $\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$, with $\int_{-\infty}^{\infty} \delta(t)dt = 1$

6.2. Unit Step signal

$\boldsymbol{u(t)}$

A step function that turns "on" at $t = 0$.

Used to model sudden changes in systems.

Mathematically: $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t \leq 0 \end{cases}$

6.3. Exponential Signal

$x(t) = e^{\alpha t}$

Describes growth $(\alpha > 0)$ or decay $(\alpha < 0)$.

Complex exponentials are used in Fourier and Laplace transforms.

General form: $x(t) = e^{(\sigma + j\omega)t} = e^{\sigma t}e^{j\omega)}$ where σ determines the amplitude decay/growth and ω represents the oscillation frequency.

6.4. Sinusoidal

$x(t) = Acos(\omega t + \varphi)$ or $x(t) = Asin(\omega t + \varphi)$

Represents periodic signals commonly found in electrical and biomedical systems.

Key parameters:

- A amplitude
- $\omega = 2\pi f$ angular frequency
- φ phase shift

6.5. Ramp signal

$r(t) = tu(t)$

A continuously increasing function used in system analysis.

Mathematically: $r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t \leq 0 \end{cases}$

6.6. Rectangular pulse

Used in digital signals and control systems.

A pulse with a specific width and amplitude.

6.7. Triangular Signal

Commonly used in modulation and waveform synthesis.

Forms a symmetric triangular shape over a given time period.

These elementary signals are essential for understanding more complex signal behaviors in communication systems, biomedical signal processing, and control engineering.

B. System

1. Definition

A **system** is a set of interrelated components that work together towards a common purpose. In various fields, particularly in biomedical, engineering, and scientific domains, systems are studied in terms of their structure, behavior, and interaction over time. In these contexts, a system typically involves a collection of variables and elements whose interactions determine the overall function and performance of the system.

2. Classification of Systems

Systems can be classified based on different criteria. In the context of **linear** and **non-linear** systems, and their **time-dependent behavior**, the classifications are as follows:

2.1. Linear Systems:

A **linear system** is one in which the relationship between the input and output is directly proportional, and the system's behavior follows the principles of superposition and scaling. This means that:

- The output is a weighted sum of the inputs.
- If you input a sum of signals into the system, the output will be the sum of the individual outputs from those signals.
- **Mathematically**: A system is linear if it satisfies the conditions of additivity and homogeneity (scaling).

Examples

- Electrical circuits with resistors, capacitors, and inductors.

- Linear differential equations describing system behavior over time.

Analysis of Linear Systems: Linear systems are easier to analyze because their behavior can be predicted using standard mathematical techniques, like Laplace transforms, Fourier analysis, and matrix operations.

2.2. Non-Linear Systems:

A **non-linear system** is one where the relationship between input and output is not proportional. In non-linear systems, small changes in input can result in large or unpredictable changes in output. This makes the system behavior more complex and difficult to analyze.

Characteristics:

- The system does not obey the principles of superposition.
- Often exhibits behaviors like bifurcations, chaos, and complex patterns.

Examples:

- Population dynamics (e.g., predator-prey models).
- Neural networks in the brain.
- Fluid dynamics.

We need more advanced techniques to analyze non-linear systems such as numerical methods (e.g., simulation, iterative techniques), chaos theory, and non-linear differential equations. Their solutions are typically more difficult to predict, and small variations in initial conditions can lead to vastly different outcomes (sensitive dependence on initial conditions).

2.3. Time-Dependent Systems (Variable in Time):

A system is **time-dependent** if its behavior changes with time. Such systems can be either linear or non-linear but their key characteristic is the dependence on time, meaning that their outputs are not constant but vary as time progresses.

Examples:

- Biological systems, like circadian rhythms, which change over the course of the day.
- Chemical reaction rates that depend on time and concentration.
- Economic models where variables like supply and demand change over time.

Analysis of time-dependent systems:

- **Time-series analysis** can be used to understand patterns and predict future behavior based on historical data.
- In differential equations, time-dependent systems are often modeled using ordinary or partial differential equations (ODEs or PDEs), which describe how the system evolves over time.

3. System modeling:

Modeling involves the creation of mathematical or computational representations of a system in order to analyze and predict its behavior. Models can be used to study both **linear** and **non-linear systems** and can take the form of equations, diagrams, or simulations.

3.1. Mathematical models

These models use equations (e.g., algebraic, differential) to describe the relationships between the system's variables.

Linear Models: Can be represented with simple linear equations.

Non-linear Models: Use non-linear equations that may involve quadratic terms, exponents, or trigonometric functions.

Examples:

- **Linear Model**: $y = mx + b$
- **Non-Linear Model**: $y = ax^2 + bx + c$ or more complex models such as $y = e^x$

3.2. Simulations

Simulations involve using computational tools to model and analyze the behavior of a system. These are particularly useful for complex, non-linear, or time-dependent systems that cannot be easily solved analytically.

Numerical simulations are often used to approximate the solutions to non-linear differential equations.

Monte Carlo simulations may be employed to model probabilistic systems.

3.3. System Identification:

System identification refers to the process of creating a model based on observed data. This process can be applied to both linear and non-linear systems, where real-world data is used to estimate the parameters of the system and validate the model's accuracy.

4. Analysis Techniques for Systems:

4.1. Linear systems analysis:

Transfer Functions: Used to represent linear systems in the frequency domain.

Laplace Transform: Used to analyze time-domain systems in the complex frequency domain.

Eigenvalues and Eigenvectors: Used to determine the stability of the system.

Fourier Transform: Helps in analyzing the frequency response of systems.

4.2. Non-Linear Systems Analysis:

Lyapunov Stability: Used to analyze the stability of non-linear systems.

Bifurcation Analysis: Examines how the system's qualitative behavior changes as a parameter is varied.

Chaos Theory: Studies the behavior of dynamic systems that are highly sensitive to initial conditions, making long-term prediction impossible in some cases.

Time-Dependent Analysis:

Differential Equations: Both ordinary differential equations (ODEs) and partial differential equations (PDEs) are used to model how systems evolve over time.

Time-Series Analysis: Techniques like autoregressive models, moving averages, and spectral analysis are used to understand and predict time-dependent systems.

5. Impulse response

5.1. Definition

The **impulse response** of a system is its output when the system is subjected to an **impulse function** (also called the Dirac delta function, denoted as δ(t)\delta(t)δ(t)) as input. Mathematically, the impulse response is the system's reaction to this idealized, instantaneous input. In the context of linear time-invariant (LTI) systems, the impulse response, h(t)h(t)h(t), characterizes the system completely, meaning that the behavior of the system for any arbitrary input can be determined using this response.

For a continuous-time system, the impulse response is the output $y(t)$ when the input $x(t)$ is $\delta(t)$ i.e. $y(t) = h(t)$ when $x(t) = \delta(t)$

In discrete-time systems, the impulse response $h[n]$ is the output when the input is $\delta[n]$, the discrete delta function.

5.2. Convolution integral:

The **convolution integral** is a mathematical operation used to determine the output of a linear time-invariant (LTI) system when the input signal is known. It describes the process of combining two functions to form a third. The output of the system is the convolution of the input signal with the system's impulse response.

For continuous-time systems, the convolution integral is defined as:

$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

Here:

$y(t)$ is the output,

$x(t)$ is the input signal,

$h(t)$ is the impulse response,

τ is the integration variable, and

$(x * h)(t)$ denotes the convolution of $x(t)$ and $h(t)$.

5.3. Convolution sum

The **convolution sum** is the discrete-time equivalent of the convolution integral. It is used to calculate the output of a discrete-time system when the input and the system's impulse response are known.

For a discrete-time LTI system, the output $y[n]$ is given by the convolution sum:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Here:

$y[n]$ is the output at discrete time n,

$x[k]$ is the input at time k,

$h[n-k]$ is the shifted impulse response at time $n-k$,

The summation runs over all values of k.

5.4. Convolution integral calculation

5.4.1. Graphical method (for continuous systems)

The **graphical method** for calculating the convolution integral is often called the **"flip and slide" method**. It involves the following steps:

Flip the Impulse Response: In the convolution integral, $h(t-\tau)$ involves a time shift, so first flip the impulse response function $h(t)$h(t) around the vertical axis.

Shift the Impulse Response: Shift the flipped impulse response by different values of t (i.e., for each value of t).

Multiply and Integrate: For each value of t, multiply the shifted impulse response by the input signal $x(\tau)$, and then integrate the result over τ.

Graphically, this means visually interpreting the overlapping area between x(τ)x(\tau)x(τ) and the flipped, shifted $h(t-\tau)$ as the convolution integral at that point

5.4.2. Analytical Method (for Continuous and Discrete Systems)

The **analytical method** involves directly performing the mathematical integration or summation of the product of the input and the shifted impulse response.

For Continuous-Time Systems: Use the convolution integral formula

$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$, and perform the integration step by step.

For Discrete-Time Systems: Use the convolution sum formula

$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$ and compute the sum for the specific range of n and k.

Both methods require knowledge of the input signal $x(t)$ (or $x[n]$) and the system's impulse response $h(t)$ (or $h[n]$).

5.5. Example calculation of the convolution integral

5.5.1. Graphical example (for continuous systems)

Suppose $x(t)$ is a rectangular pulse function, and $h(t)$ is a simple exponential decay. The process involves flipping $h(t)$, shifting it over different time intervals, and multiplying it with the input signal to find the overlapping area at each point. The integral would then calculate the exact area under the curve of the product, giving the output $y(t)$.

5.5.2. Analytical example (for continuous systems)

If $x(t) = e^{-t}$ for $t \geq 0$ and h(t)=1 for $t \geq 0$, the convolution integral is:

$$y(t) = \int_0^t e^{-\tau}xd\tau = 1 - e^{-t}$$

5,53. Discrete Example

For a discrete-time input $x[n] = \{1,2,3\}$ and impulse response $h[n] = \{1,1,1\}$, the convolution sum is:

$y[0] = x[0]h[0] = 1 \times 1 = 1$

$y[1] = x[0]h[1] + x[1]h[0] = 1 \times 1 + 2 \times 1 = 3$

$y[2] = x[0]h[2] + x[1]h[1] + x[2]h[0] = 1 \times 1 + 2 \times 1 + 3 \times 1 = 6$

So, the output sequence is $y[n] = \{1,3,6\}$.

Conclusion:

In biomedical and other scientific fields, understanding the behavior of systems is essential, whether they are linear or non-linear and time-dependent or time-independent. The impulse response of a system plays a crucial role in this understanding, as it allows us to analyze how the system reacts to various inputs. The convolution integral (or sum) serves as a powerful tool for calculating the output of a linear time-invariant (LTI) system in response to any given input.

To analyze these systems effectively, we employ various mathematical and computational techniques, including differential equations, system identification, and numerical simulations. Both graphical and analytical methods for computing convolutions exist, with the graphical method often providing a more intuitive understanding, while the analytical method offers greater precision. By modeling and analyzing both linear and non-linear systems, we can gain insights into their behavior, predict future states, and develop solutions to real-world problems in the biomedical field and beyond.

Chapter 2: Fourier analysis of continuous-time signals

Introduction

Fourier analysis is a mathematical technique used to break down a continuous-time signal into its constituent sinusoids (sine and cosine functions) of various frequencies. This technique is fundamental in signal processing, communication systems, and many other fields of engineering and physics. Fourier analysis helps in understanding the frequency content of a signal, allowing us to analyze, modify, and synthesize signals more easily.

1. Fourier transform (FT)

The **Fourier Transform (FT)** is a mathematical operation that transforms a time-domain signal (continuous-time signal) into its frequency-domain representation. This transformation expresses the signal as a sum of sinusoids of different frequencies, amplitudes, and phases.

The continuous-time Fourier transform (CTFT) of a signal x(t)x(t)x(t) is given by:

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

Where:

$X(f)$ is the frequency-domain representation of the signal

$x(t)$ is the continuous-time signal in the domain

f is the frequency variable

j is the imaginary unit

The Fourier transform essentially converts a signal from the time domain (which represents how the signal evolves over time) to the frequency domain (which represents how the signal is composed of different frequencies).

2. Inverse Fourier transform

The **Inverse Fourier Transform** allows us to reconstruct the original time-domain signal from its frequency-domain representation. It is given by:

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi tf}df$$

This equation tells us that by summing all the sinusoids (each corresponding to a different frequency) in the frequency domain, we can obtain the original time-domain signal.

3. Properties of the Fourier transform

Some important properties of the Fourier transform include:

Linearity: If $x_1(t)$ and $x_2(t)$ are two signals, then the Fourier transform of

$ax_1(t) + bx_2(t)$+bx2(t) is $aX_1(f) + bX_2(f)$.

Time shifting: If the signal is shifted by t_0 in time, the Fourier transform is multiplied by $e^{-j2\pi f t_0}$.

Frequency shifting: If the signal is multiplied by $e^{j2\pi t f_0}$ in the time domain, the Fourier transform is shifted by f_0 in the frequency domain.

Convolution: The Fourier transform of the convolution of two signals is the product of their Fourier transforms.

Parseval's Theorem: The total energy of a signal in the time domain is equal to the total energy in the frequency domain.

4. Fourier series (Periodic signals)

For periodic signals, **Fourier series** is used instead of the Fourier transform. A **Fourier series** represents a periodic signal as a sum of sines and cosines (or complex exponentials). If $x(t)$ is a periodic signal with period T, its Fourier series representation is:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_0 t}$$

Where:

$C_n = \frac{1}{T}\int_0^T x(t)e^{-j2\pi n f_0 t}dt$ are the Fourier coefficients (complex numbers) that represent the amplitude and phase of the n-th harmonic.

$f_0 = \frac{1}{T}$ is the fundamental frequency.

The sum runs over all harmonics of the signal.

The Fourier series is particularly useful for analyzing periodic signals, as it allows us to express the signal in terms of its frequency components.

5. Applications of Fourier analysis

Fourier analysis has broad applications in several areas, including:

Signal Processing: Used to filter, compress, and analyze signals. For example, audio signals can be transformed into the frequency domain to filter out unwanted noise.

Communications: In communications, Fourier transforms are used to modulate and demodulate signals, enabling efficient data transmission.

Spectral Analysis: Fourier analysis is used to analyze the frequency content of signals, such as identifying the dominant frequencies in a speech signal or a music waveform.

Image Processing: Fourier transforms are used to analyze and manipulate images in the frequency domain, helping in tasks like image compression or enhancement.

6. Example of Fourier Transform

Consider the simple example of a rectangular pulse signal $x(t)$ that is nonzero between $t = -\frac{T}{2}$ and $t = T/2$:

$$x(t) = \begin{cases} 1, if & |\, t \,| \leq T/2 \\ 0, & otherwise \end{cases}$$

To compute the Fourier transform of this signal, we would use the formula:

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

Since the signal is a rectangular pulse, the Fourier transform will result in a **sinc function** in the frequency domain:

$$X(f) = T \cdot sinc(fT)$$

Where $sinc(x) = \frac{sin(\pi x)}{\pi x}$. This result shows how the time-domain signal (a rectangular pulse) corresponds to a sinc-shaped spectrum in the frequency domain.

7. Representation of signals in terms of orthogonal basis functions:

In signal processing and functional analysis, a signal can be represented as a **linear combination** of basis functions. If the basis functions are **orthogonal**, the coefficients in the expansion can be easily computed and the signal's representation becomes very efficient.

7.1. Orthogonal Basis Functions:

Orthogonal basis functions are a set of functions that are mutually perpendicular, meaning the inner product (or integral) of any two distinct basis functions is zero. In mathematical terms, two functions $\phi_1(t)$ and $\phi_2(t)$ are orthogonal if:

$\int_{-\infty}^{\infty} \phi_1(t)\phi_2(t)\, dt = 0$ for $\phi_1(t) \neq \phi_2(t)$

These basis functions can be used to express any signal within a given space as a sum of these orthogonal functions. The most well-known example of orthogonal basis functions is the **Fourier series**, where sines and cosines form the orthogonal basis for periodic signals.

7.2. Representation of a signal:

For a signal $x(t)$, it can be expressed in terms of a set of orthogonal basis functions $\{\phi n(t)\}$ as:

$$x(t) = \sum_{n=0}^{\infty} C_n \phi_n(t)$$

Where:

C_n are the coefficients to be determined.

$\phi_n(t)$ are the orthogonal basis functions.

In practice, these basis functions are often chosen such that they have specific properties, such as being periodic (like the sine and cosine functions in Fourier series) or being defined over a specific domain.

7.3. Computing the coefficients:

The coefficients C_n represent the contribution of each basis function to the signal. For orthogonal bases, the coefficients can be computed using the **inner product** of the signal $x(t)$ and each basis function $\phi_n(t)$.

For continuous signals, the coefficient C_n is given by: $C_n = \frac{\int_{-\infty}^{\infty} x(t)\phi_n(t)dt}{\int_{-\infty}^{\infty} \phi_n^2(t)dt}$

If the basis functions are normalized, the denominator becomes 1, simplifying the formula to:

$$C_n = \int_{-\infty}^{\infty} x(t)\phi_n(t)dt$$

For discrete signals, the sum replaces the integral:

$$C_n = \sum_{n=0}^{N-1} x[n]\phi_n[n]$$

7.4. Minimal Quadratic Error (Least Squares Approach):

When trying to approximate a signal x(t)x(t)x(t) using a finite number of basis functions, there will typically be some error because the signal is infinite-dimensional, and we cannot represent it with just a finite set of basis functions. The goal is to minimize this error, and the method commonly used is called the **least squares approximation**.

The **least squares method** minimizes the error between the signal and its approximation by finding the best-fit coefficients. The error is typically defined as the **squared error** (or residual) between the original signal $x(t)$ and its approximation $\hat{x}(t)$:

$$E = \int_{-\infty}^{\infty} (x(t) - \hat{x}(t))^2 \, dt$$

Where $\hat{x}(t)$ is usually expressed in terms of the chosen basis functions as:

$$\hat{x}(t) = \sum_{n=0}^{N-1} C_n \phi_n(t)$$

To minimize the error, the coefficients C_n are found by solving for the values that minimize the integral of the squared error:

$$E = \int_{-\infty}^{\infty} \left(x(t) - \sum_{n=0}^{N-1} C_n \phi_n(t) \right)^2 dt$$

This minimization is usually done by taking the derivative of the error with respect to each coefficient C_n and setting it equal to zero. The result is a set of **normal equations** that can be solved to find the optimal coefficients C_n for the approximation.

7.5. Solution to the least squares problem:

The solution to the least squares problem can be represented as a system of equations:

$$\sum_{n=0}^{N-1} C_n \left(\int_{-\infty}^{\infty} \phi_m(t) \phi_n(t)\, dt \right) = \int_{-\infty}^{\infty} x(t) \phi_m(t)\, dt$$

This system is often written in matrix form as:

$$Ac = b$$

Where:

A is a matrix whose elements are the inner products of the basis functions $\phi_n(t)$

c is the column vector of coefficients C_n,

b is the column vector of inner products between $x(t)$ and $\phi_m(t)$

In practical applications, this system can be solved using numerical methods such as **Gaussian elimination** or **matrix inversion**.

7.6. Example:

If we want to approximate a signal $x(t)$ using a set of **Fourier basis functions** (sines and cosines), we can compute the Fourier coefficients C_n using the Fourier series approach. These coefficients are obtained by calculating the inner product between $x(t)$ and the corresponding sine or cosine function.

For instance, for a continuous-time periodic signal x(t)x(t)x(t), the Fourier coefficients are:

$$C_n = \frac{1}{T}\int_0^T x(t)e^{-j2\pi n f_0 t}dt$$

This approach provides an approximation of x(t)x(t)x(t) using a finite number of Fourier components, and the error can be minimized in the least squares sense by selecting the appropriate number of terms in the series.

8. Fourier series representation: sinusoidal and exponential forms

The Fourier series allows us to represent a periodic signal as a sum of sinusoidal or exponential functions. There are two main representations:

8.1. Trigonometric (sinusoidal) form

A periodic function $x(t)$with period T is useful for real-valued periodic signals in practical applications, such as electrical engineering and signal processing is expressed as a sum of sines and cosines:

$$x(t) = a_0 \sum_{n=1}^{\infty} [a_n cos(n\omega_0 t) + b_n sin(n\omega_0 t)]$$

Where:

$\omega_0 = \frac{2\pi}{T}$ is the fundamental angular frequency

$a_0 = \frac{1}{T}\int_0^T x(t)dt$ is the DC component frequency, (DC stands for dot product)

$a_n = \frac{2}{T}\int_0^T x(t)cos(n\omega_0 t)dt$ and $b_n = \frac{2}{T}\int_0^T x(t)sin(n\omega_0 t)dt$ are the Fourier coefficients

8.2. Exponential (complex) form

The Fourier series can also be expressed using complex exponentials:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

where $C_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}$ is the Fourier coefficients.

$e^{jn\omega_0 t}$ represents sinusoidal oscillations in complex form.

The coefficients C_n contain both amplitude and phase information.

The exponential form is particularly useful for mathematical analysis and signal processing, as it simplifies differentiation and integration in frequency-domain analysis.

Comparison of the Two Forms

Representation	Formula	Used for
Sinusoidal Form	$x(t) = a_0 \sum_{n=1}^{\infty} [a_n cos(n\omega_0 t) + b_n sin(n\omega_0 t)]$	Real-valued periodic signals
Exponential Form	$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$	Mathematical and frequency analysis

Both forms are equivalent and can be converted from one to another

8.3. Complex Fourier spectrum

The Complex Fourier Spectrum refers to the representation of a signal in the frequency domain using the Fourier Transform or Fourier Series in exponential form. It describes the amplitude and phase of the frequency components that make up a signal.

8.3.1. Definition

A signal x(t)x(t)x(t) can be expressed in terms of its frequency components using the Fourier Transform: $X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$

where:

$X(f)$ is the complex Fourier spectrum of the signal, it contains both magnitude and phase

The inverse Fourier transform is $x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} dt$

8.3.2. Interpretation of the complex Fourier spectrum

The spectrum $X(f)$ is complex, meaning it has both real and imaginary parts.

It can be written as: $X(f) = \mid X(f) \mid e^{jarg(X(f))}$

where:

$\mid X(f) \mid$ is the magnitude spectrum, showing the strength of each frequency.

$arg(X(f)$ is the phase spectrum, showing the phase shift at each frequency.

8.3.3. Example: Fourier transform of a sinusoidal signal

If $x(t) = Acos(2\pi f_0 t)$, its Fourier Transform is: $X(f) = \frac{A}{2}[\delta(f - f_0) + \delta(f + f_0)]$

The magnitude spectrum shows two spikes at $\pm f_0$.

The phase spectrum indicates the phase shift of each component.

8.3.4. Visualization of the complex spectrum

The Fourier spectrum is typically visualized as two plots:

Magnitude spectrum $|X(f)|$ – Shows the intensity of frequency components.

Phase spectrum $arg(X(f)$ – Shows the phase shift of each frequency component.

9. Power spectral density (PSD)

The Power spectral density (PSD) represents how the power of a signal is distributed across different frequencies. It provides insights into the frequency components of a signal and is commonly used in signal processing, communications, and biomedical engineering (e.g., Electrocardiography (ECG), electromyography (EMG), and electroencephalography (EEG) analysis).

9.1. Definition

The Power Spectral Density (PSD) of a signal $x(t)$ is defined as:

$$S_x(f) = \lim_{T \to +\infty} \frac{1}{T} |X_T(f)|^2$$

Where:

$X_T(f)$ is the Fourier transform of the signal over a period time T

$S_x(f)$ gives the power per unit frequency in W/Hz or dB/Hz

For a discrete-time signal $x[n]$, the PSD is often estimated using the periodogram or the Welch method.

9.2. Relationship with the autocorrelation function

The Wiener-Khinchin theorem states that the PSD is the Fourier transform of the autocorrelation function $R_x(\tau)$:

$$S_x(f) = \int_{-\infty}^{\infty} R_x(\tau)\, e^{-j2\pi f\tau} d\tau = \int_{-\infty}^{\infty} E[x(t)x(t+\tau)]\, e^{-j2\pi f\tau} d\tau$$

This means that PSD provides frequency-domain information about signal variations over time.

9.3. Applications of PSD

- **Biomedical Engineering**: Analyzing EEG (brain signals), ECG (heart signals), and EMG (muscle signals).
- **Communications**: Studying noise and bandwidth of signals in wireless networks.
- **Vibration Analysis**: Monitoring mechanical systems to detect faults.

9.4. PSD Estimation methods

To estimate PSD, common techniques include:

- **Periodogram**: Direct computation of squared Fourier transform.
- **Welch's method**: Averages multiple periodograms for a smoother estimate.
- **Burg method**: Uses autoregressive (AR) modeling.

9.5. MATLAB example: estimating PSD

You can use MATLAB to compute and visualize the PSD of a signal:

```
fs = 1000; % Sampling frequency (Hz)
t = 0:1/fs:1-1/fs; % Time vector
x = sin(2*pi*50*t) + 0.5*randn(size(t)); % Signal (50 Hz sine wave + noise)

% Compute PSD using Welch's method
[pxx, f] = pwelch(x, hamming(256), 128, 256, fs);

% Plot PSD
figure;
plot(f, 10*log10(pxx)); % Convert to dB scale
xlabel('Frequency (Hz)');
ylabel('Power/Frequency (dB/Hz)');
title('Power Spectral Density (PSD)');
grid on;
```

10. Convergence of Fourier series, Gibbs phenomenon, and Fourier transform properties

10.1. Convergence of Fourier series

A Fourier series represents a periodic function as a sum of sine and cosine functions. The convergence of the series depends on the function's properties.

Dirichlet conditions for convergence

A Fourier series converges to a function $f(x)$ if:

$f(x)$ is absolutely integrable over one period: $\int_{-\pi}^{\pi}|f(x)|dx < \infty$

$f(x)$ has a finite number of discontinuities in any finite interval.

$f(x)$ has a finite number of maxima and minima in any finite interval.

If these conditions hold, the Fourier series converges pointwise to $f(x)$ except at discontinuities, where it converges to the average of the left-hand and right-hand limits.

10.2. Gibbs Phenomenon

The Gibbs phenomenon occurs when a Fourier series approximates a function with a discontinuity**.** Near the discontinuity, the Fourier series overshoots the true function, creating oscillations that do not disappear as more terms are added.

Key Characteristics:

- The overshoot is about 9% of the jump size, regardless of how many terms are included.
- Increasing the number of terms reduces the width of the oscillations but does not eliminate the overshoot.
- It is significant in signal processing, where truncation of Fourier series affects accuracy.

MATLAB example of Gibbs phenomenon:

```
x = linspace(-pi, pi, 1000);
f = sign(sin(x)); % Square wave signal
N = 10; % Number of Fourier terms

y = zeros(size(x));
for k = 1:N
    y = y + (1/(2*k-1)) * sin((2*k-1)*x);
end
y = (4/pi) * y; % Fourier series approximation

plot(x, f, 'k', 'LineWidth', 2); hold on;
plot(x, y, 'r', 'LineWidth', 1.5);
xlabel('x'); ylabel('Function Value');
legend('Original Square Wave', 'Fourier Approximation');
title('Gibbs Phenomenon');
grid on;
```

10.3. Fourier transform properties

The Fourier transform (FT) converts a signal from the time domain to the frequency domain and is defined as: $F(f) = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi ft} dt$

Properties of Fourier Transform:

Linearity**:** $F[af_1(t) + bf_2(t)] = aF_1(f) + bF_2(f)$

Time Shifting**:** $F[f(t - t_0)] = e^{-j2\pi ft_0} F(f)$

Frequency Shifting**:** $F\left[e^{j2\pi tf_0} f(t)\right] = F(f - f_0)$

Scaling Property**:** $F[f(at)] = \frac{1}{|a|} F\left(\frac{f}{a}\right)$

Parseval's Theorem $\int_{-\infty}^{\infty} |f(t)|^2\, dt = \int_{-\infty}^{\infty} |F(f)|^2\, df$

Convolution Theorem**:** $F[f_1(t) * f_2(t)] = F_1(f)F_2(f)$: Convolution in time domain is multiplication in frequency domain

11. Fourier transform of singular functions

A singular function is a function that exhibits discontinuities, impulses, or other non-smooth behaviors. Some important singular functions in signal processing include the Dirac delta function**,** step function, and other piecewise discontinuous functions.

11.1. Fourier Transform of the Dirac Delta Function $\delta(t)$

The Dirac delta function $\delta(t)$ is defined as: $\delta(t) = \begin{cases} \infty, t = 0 \\ 0, t \neq 0 \end{cases}$ with the property: $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

Fourier transform of $\delta(t)$

$$F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi ft} dt$$

Using the sifting property of the delta function: $F[\delta(t)] = 1$

This means the Fourier Transform of δ(t)\delta(t)δ(t) is a constant across all frequencies.

Fourier Transform of $\delta(t - t_0)$ (Shifted Delta Function)

$F[\delta(t - \boldsymbol{t_0})] = e^{-j2\pi f\boldsymbol{t_0}}$ which represents a phase shift in the frequency domain.

11.2. Fourier transform of the unit step Function $u(t)$

The Heaviside step function is defined as: $u(t) = \begin{cases} 1\ t \geq 0 \\ 0, t < 0 \end{cases}$

Fourier transform of $u(t)$

$F[u(t)] = \int_0^{\infty} e^{-j2\pi ft}\, dt = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$ which shows that the Fourier transform of the unit step function consists of a principal value $\frac{1}{j2\pi f}$ and DC component $\frac{1}{2}\delta(f)$

11.3. Fourier Transform of the Sign Function $sgn(t)$

The sign function is defined as: $u(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$

Fourier transform of $sgn(t)$

Using the integral definition of the Fourier transform: $F[sgn(t)] = \int_{-\infty}^{\infty} sgn(t) e^{-j2\pi ft}\, dt = \frac{2}{j2\pi f} = \frac{1}{j\pi f}$ which is **purely imaginary** and represents a Hilbert transform in the frequency domain**.**

11.4. Summary of Fourier transforms of singular functions

Function $\boldsymbol{f(t)}$	$\delta(t)$	$\boldsymbol{\delta(t - t_0)}$	$u(t)$	$sgn(t)$
Fourier Transform $\boldsymbol{F(f)}$	1	$e^{-j2\pi f t_0}$	$\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$	$\frac{1}{j\pi f}$

12. Energy spectral density (ESD)

The Energy spectral density (ESD**)** describes how the energy of a signal is distributed across different frequencies. It is particularly useful for analyzing finite-energy signals (also called energy signals), which have a total energy given by:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

12.1. Definition

The Energy spectral density is defined as the squared magnitude of the Fourier transform of a signal $x(t)$:

$S_E(f) = |x(f)|^2$ where $X(f)$ is the Fourier transform of $x(t)$:

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft}\, dt$$

Thus, the total energy of the signal can be expressed in terms of the Energy Spectral Density:

$$E = \int_{-\infty}^{\infty} S_E(f)df$$

This follows from Parseval's theorem, which states that the energy computed in the time domain is equal to the energy computed in the frequency domain.

12.2. Key properties of energy spectral density

Non-Negative: $S_E(f) \geq 0, \forall\, f$, since it is the squared magnitude of the Fourier Transform.

The integral of the energy spectral density over all frequencies gives the total signal energy: $E = \int_{-\infty}^{\infty} S_E(f)df$

For real-valued signals, the energy spectral density is symmetric: $S_E(f) = S_E(-f)$

12.3. Example: energy spectral density of a rectangular pulse

Consider a rectangular pulse signal: $x(t) = \begin{cases} 1, |t| \leq T/2 \\ 0, otherwise \end{cases}$

Its Fourier transform is: $X(f) = Tsinc(Tf)$

Thus, the Energy spectral density is: $S_E(f) = |X(f)|^2 = |T|^2 sinc^2(Tf)$

This shows that the energy is concentrated around low frequencies.

12.4. Comparison with power spectral density (PSD)

Energy Spectral Density (ESD) applies to energy signals (finite energy, zero average power).

Power Spectral Density (PSD) applies to power signals (infinite energy, finite average power).

For a stationary signal $x(t)$ with infinite energy, we use the Power Spectral Density (PSD), which is related to the autocorrelation function via the Wiener-Khinchin theorem.

13. Fourier series of discrete-time signals: complex exponentials and harmonic relations

13.1. Introduction to discrete-time Fourier series (DTFS)

The Fourier Series representation for discrete-time periodic signals expresses a periodic signal as a sum of complex exponentials that are harmonically related.

For a discrete-time periodic signal $x[n]$ with period N, the discrete-time Fourier series (DTFS) is given by: $x[n] = \sum_{k=0}^{N-1} C_k e^{j\frac{2\pi}{N}kn}$ where the Fourier coefficients $C_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$ determine how much of each complex exponential component contributes to the original signal.

13.2. Steps for Determining DTFS of a Discrete-Time Signal

Identify the period N: Ensure that the given signal is periodic with a period NNN.

Compute the Fourier coefficients C_k: Use the formula to find the coefficients by summing over one period.

Reconstruct the signal: Using the inverse DTFS formula, sum the frequency components to obtain $x[n]$.

Analyze the frequency spectrum: The coefficients C_k describe how much of each frequency component is present in the signal.

13.3. Properties of the DTFS

Periodicity of Coefficients: The Fourier coefficients $C_k$are periodic with period N, meaning $C_{k+N} = C_k$.

Orthogonality of Basis Functions: The exponentials $e^{j\frac{2\pi}{N}kn}$ form an orthogonal basis, allowing the signal to be decomposed uniquely into these components.

Parseval's Theorem: The total energy of the signal is preserved in the frequency domain:

13.4. Examples: DTFS of a simple discrete-time signal

Example 1

Consider a periodic signal: $x[n] = e^{j\frac{2\pi}{N}n}$ which is itself a harmonic complex exponential. The Fourier series coefficients for this signal are: $C_k = \begin{cases} 1, k = 1 \\ 0, k \neq 1 \end{cases}$ indicating that the signal consists of only one frequency component.

Example 2

Consider a periodic signal: $x[n] = 1, \ \ 0 \leq n < N$ For this signal, the Fourier coefficients are calculated as: $C_k = \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}kn}$which results in a sinc-like frequency spectrum.

13.5. Harmonic relation of complex exponentials

The fundamental frequency of a discrete-time periodic signal with period N is: $\omega_0 = \frac{2\pi}{N}$

Each term in the Fourier series expansion corresponds to a harmonic of the fundamental frequency: $e^{jk\omega_0 n} = e^{j\frac{2\pi}{N}kn}$ where k is an integer representing different frequency components (harmonics) in the signal.

14. Discrete-time Fourier transform (DTFT)

The Discrete-Time Fourier Transform (DTFT) is used to analyze the frequency content of discrete-time signals. Unlike the **Discrete Fourier Series (DFS)**, which is used for periodic signals, the DTFT applies to aperiodic signals and provides a continuous frequency spectrum.

14.1. Definition

For a discrete-time signal $x[n]$, the DTFT is defined as:

$$X(e^{-j\omega n}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Where:

$X(e^{-j\omega n})$ is the frequency representation of $x[n]$ as a complex-valued function of frequency

ω is the normalized angular frequency in radians, with $-\pi \leq \omega \leq \pi$

The summation extends over all discrete time indices n.

14.2. Inverse DTFT

To recover $x[n]$ from its DTFT, the inverse DTFT is given by:

$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})X(e^{j\omega n})\, d\omega$

Which reconstructs the original time-domain signal from its frequency representation.

14.3. Properties of the DTFT

Some key properties of the DTFT include:

Linearity: $ax_1[n] + bx_2[n] \leftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$

Time Shifting: $x[n - n_0] \leftrightarrow X(e^{-j\omega n_0})X(e^{j\omega})$

Frequency Shifting: $e^{j\omega n_0}x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$

Convolution Property: : $x_1[n] * bx_2[n] \leftrightarrow X_1(e^{j\omega})X_2(e^{j\omega})$

Parseval's Theorem (Energy conservation): $\sum_{n=-\infty}^{\infty}|x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|X(e^{j\omega})|^2 d\omega$

14.4. Example: DTFT of a finite-length signal

Consider the discrete-time signal: $x[n] = \{1,2,3,4\}, 0 \leq n \leq 3$

Its DTFT is: $X(e^{j\omega}) = 1 + 2e^{-j\omega} + 3e^{-j2\omega} + 4e^{-j3\omega}$ which describes how the frequency components are distributed in the signal.

14.5. Relationship to Other Transforms

DTFT vs. DFT: The **Discrete Fourier Transform (DFT)** is a sampled version of the DTFT, used for numerical computations.

DTFT vs. Z-Transform: The DTFT is a special case of the **Z-Transform** when evaluated on the unit circle $z = e^{j\omega}$.

15. Fourier Transform of periodic signals

For periodic signals, the Fourier Transform does not exist in the conventional sense because periodic signals have an infinite duration. Instead, periodic signals are represented using the **Fourier Series**, and their Fourier Transform leads to a discrete spectrum.

15.1. Representation of a periodic signal

A continuous-time periodic signal $x(t)$ with fundamental period T_0 can be expressed using the Fourier Series as: $x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$

where:

$\omega_0 = \frac{2\pi}{T_0}$ is the fundamental angular frequency.

$C_k = \frac{1}{T_0}\int_0^T x(t)e^{-jk\omega_0 t}dt$ are the Fourier series coefficients

15.2. Fourier transform of a periodic signal

The Fourier Transform of a periodic signal consists of **impulses (Dirac delta functions)** at the harmonics of the fundamental frequency ω_0: $X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} C_k \delta(\omega - k\omega_0)$

where:

$\delta(\omega - k\omega_0)$ represents impulses at discrete frequencies $k\omega_0$.

The amplitude of each impulse is proportional to C_k.

This shows that a periodic signal in the time domain corresponds to a discrete set of frequency components (harmonics) in the frequency domain.

15.3. Example: Fourier transform of a square wave

Consider a square wave signal $x(t)$ with period T_0 and duty cycle of 50%. Its Fourier series representation is: $x(t) = \sum_{k=-\infty}^{\infty} \frac{\sin(k\pi/2)}{k\pi/2} e^{jk\omega_0 t}$

Its Fourier transform consists of impulses at multiples of ω_0, weighted by the coefficients C_k.

Notes

The Fourier Transform of a periodic signal is a **sum of delta functions** at the harmonic frequencies.

The spacing between spectral lines is determined by the fundamental frequency ω_0.

The magnitude of each spectral component is given by the Fourier series coefficients C_k.

16. Laplace transform

The Laplace transform of a function $f(t)$, denoted $L\{f(t)\}$, is defined as:

$$L\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt$$

where:

$t \geq 0$ (assuming $f(t)$ is defined for $t \geq 0$,

$s = \sigma + j\omega$ (a complex number, where σ and ω are real, and j is the imaginary unit).

16.1. Region of convergence (ROC)

The Region of convergence (ROC) is the set of values of s for which the Laplace transform integral converges. This region depends on the nature of the function $f(t)$. For example:

For exponentially decaying functions, the ROC is a left half-plane in the complex plane.

For impulsive functions, the ROC could be the entire real line or a central region in the complex plane.

16.2. Unilateral Laplace transform

The **Unilateral Laplace transform** is a specific version of the Laplace Transform, where the function $f(t)$ is defined only for $t \geq 0$. This simplifies the transform, as it involves only the positive time axis (for ≥ 0) and does not require consideration of negative values of t.

16.2. Relationship between the Laplace transform and the Fourier transform

The Fourier transform is a special case of the Laplace Transform, where the complex variable s is purely imaginary. Specifically, when $s = j\omega$, the Laplace transform becomes the Fourier transform.

Formally:

$$L\{f(t)\}\,|_{s=j\omega} = F\{f(t)\}$$

This means that:

$$F\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

In the case of the Laplace transform, the integral is taken over $[0, \infty)$, while the Fourier ransform typically integrates over $(-\infty, \infty)$.

16.3. Differences between the Laplace transform and the Fourier transform:

Integration domain: The Laplace transform integrates from $0\ to\ \infty$, whereas the Fourier transform integrates over the entire real line $(-\infty, \infty)$.

Complex parameter: The Laplace transform uses a complex parameter $s = \sigma + j\omega$ s, while the Fourier transform uses, meaning the variable is purely imaginary in the case of Fourier.

Applications: The Laplace Transform is more general and is particularly useful for solving differential equations with initial conditions. The Fourier Transform is more specific to analyzing stationary and periodic signals.

In summary, the Laplace Transform is more general, while the Fourier Transform is a specific form of the Laplace Transform, used mainly for analyzing signals and systems in the frequency domain.

17. Z-Transform

The Z-Transform is a powerful tool used in signal processing and control systems to analyze discrete-time signals and systems. It is defined as:

$$Z\{x[n]\} = X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

where:

$x[n]$ is the discrete-time signal (a sequence of values),

z is a complex variable, and z=rejθz = r e^{j\theta}z=rejθ (polar form),

n is the discrete-time index (typically starting from $n = 0$).

17.1. Region of convergence (ROC)

The Region of convergence (ROC) is the set of values of z for which the Z-Transform converges. It depends on the nature of the sequence $x[n]$. The ROC is crucial for analyzing the stability and behavior of systems. Some examples:

If $x[n]$ is absolutely summable, the ROC may include a circle centered at the origin in the complex plane.

If $x[n]$ is a decaying exponential, the ROC may be an annular region (a ring-shaped area in the complex plane).

The ROC provides insights into the stability of discrete-time systems, and different sequences have different regions of convergence.

17.2. Inverse Z-Transform

The Inverse Z-Transform is used to recover the original time-domain sequence $x[n]$ from its Z-Transform $X(z)$. It can be computed using several methods:

Partial Fraction Expansion: This method involves decomposing $X(z)$ into simpler fractions, which can then be inverted using known inverse Z-Transforms.

Contour Integration: This approach uses complex integration around a closed contour in the complex plane to recover $x[n]$.

Power Series Expansion: This method involves expanding $X(z)$ as a power series and determining the coefficients of the series.

17.3. Division method

The Division method (or long division) is a technique for finding the inverse Z-Transform. It involves dividing the Z-Transform $X(z)$ by a known function to express $X(z)$ in a form that is easier to invert. This method is particularly useful when $X(z)$ has a rational form (i.e., a ratio of polynomials in z).

Steps for the division method:

Express $X(z)$ as a ratio of polynomials $(z) = \frac{P(z)}{Q(z)}$.

Perform polynomial long division on $P(z)$ and $Q(z)$.

Once the division is done, use known inverse Z-Transforms for the resulting terms to find $x[n]$.

17.4. Relation between Z-Transform and Laplace transform

The Z-Transform and the Laplace transform are closely related, as both are used to analyze signals and systems in the complex plane. The Z-Transform is primarily used for discrete-time signals, while the Laplace Transform is used for continuous-time signals. The key relationship is:

The **Z-Transform** is often considered as the discrete-time counterpart of the Laplace Transform.

If $s = j\omega$ (purely imaginary), the Laplace Transform reduces to the Fourier transform.

For discrete systems, the Z-Transform can be thought of as the discrete counterpart to the Laplace Transform, and they are connected through the concept of sampling: the Z-Transform arises by discretizing the continuous-time Laplace Transform using the sampling period.

More specifically, for a discrete-time system, the Z-Transform of a signal can be derived from the Laplace Transform by discretizing the Laplace Transform, replacing the continuous-time variable sss with the discrete-time variable zzz. The Z-Transform is the discrete analog of the Laplace Transform, and the relationship between them is important for transitioning between discrete and continuous domains.

Conclusion

Fourier analysis serves as a powerful tool for understanding and manipulating signals by transforming them from the time domain to the frequency domain. This transformation enables us to gain valuable insights into the frequency content of signals, which proves useful in various applications, including signal processing, communications, and spectral analysis. Techniques such as the Fourier transform and Fourier series allow us to express both periodic and non-periodic signals in terms of their frequency components. Additionally, representing signals using orthogonal basis functions enhances our ability to analyze and approximate them efficiently. Computing the coefficients of these basis functions is crucial, and when a signal cannot be perfectly represented by a finite set of basis functions, the least squares method offers a way to minimize the error between the actual signal and its approximation. This approach is widely applicable in signal processing, data compression, and other fields requiring effective signal approximation and reconstruction. Together, these methods and techniques provide a comprehensive framework for analyzing and manipulating signals across various domains. In addition The Z-Transform is used for discrete-time signals and systems, with a Region of Convergence (ROC) that plays a significant role in analyzing system behavior. The Inverse Z-Transform allows recovery of the time-domain sequence using techniques like partial fraction expansion or long division. The Z-Transform is closely related to the Laplace Transform, acting as its discrete-time counterpart.

Chapter 3: Image Filtering, enhancement, and restoration

1. Image Filtering

1.1. Gaussian Filter

Definition

The Gaussian filter is a widely used filter in image processing for reducing noise and smoothing images. It is defined mathematically as:

$$G(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

where:

$G(x,y)$ is the value of the Gaussian filter at coordinates (x,y).

σ is the standard deviation of the Gaussian distribution, which controls the amount of smoothing.

The term $2\pi\sigma^2$ normalizes the filter so that the sum of all coefficients equals 1.

1.2. Properties

Smoothing: The Gaussian filter smooths an image by averaging pixel values with their neighbors in a weighted manner, where pixels closer to the center have a higher weight.

Isotropic: The filter is isotropic, meaning it has the same effect in all directions.

Controlled Blurring: The degree of blurring can be controlled by adjusting σ. A larger σ results in more smoothing.

1.3. Example

Applying a Gaussian Filter**:**

Consider a 5x5 kernel for a Gaussian filter with σ=1:

$$\begin{bmatrix} 0.0613 & 0.1247 & 0.0613 \\ 0.1247 & 0.2486 & 0.1247 \\ 0.0613 & 0.1247 & 0.0613 \end{bmatrix}$$

When this kernel is convolved with an image, each pixel in the output image is computed as the weighted sum of the neighboring pixels based on the kernel values. This results in a smoother image with reduced noise.

To visualize the effect of a Gaussian filter, consider a noisy image. After applying the filter, the image appears smoother, and the noise is less pronounced.

2. Image enhancement

2.1. Contrast stretching

Contrast stretching is a technique used to enhance the contrast of an image by expanding the range of intensity values. The transformation can be expressed mathematically as:

$$I_{enhanced}(x,y) = \frac{I(x,y) - I_{min}}{I_{max} - I_{min}} \times (L-1)$$

where:

$I_{enhanced}(x,y)$ is the enhanced pixel value at coordinates (x,y).

$I(x,y)$ is the original pixel value at coordinates (x,y).

I_{min} and I_{max} are the minimum and maximum pixel values in the original image.

L is the number of intensity levels (e.g., 256 for an 8-bit image).

2.2. Steps for contrast stretching

Find minimum and maximum: Determine I_{min} and I_{max} from the original image.

Apply transformation: Use the contrast stretching formula to compute the new pixel values.

Resulting image: The resulting image will have its pixel values spread across the full range, enhancing the visual contrast.

2.3. Example

Let's consider an image with pixel values ranging from 50 to 200.

Set $I_{min} = 50$ and $I_{max} = 200$.

For a pixel value $I(x,y) = 100$:

$$I_{enhanced}(x,y) = \frac{100 - 50}{200 - 100} \times (255 - 1) \approx 84.67$$

This transformation adjusts the pixel value from 100 to approximately 85, effectively stretching the contrast. We can note that:

Before Contrast Stretching: The image may appear dull and lacking detail.

After Contrast Stretching: The image shows improved contrast, making features more distinguishable.

By applying the Gaussian filter and contrast stretching techniques, images can be significantly enhanced for better visual quality and analysis. These methods are foundational in various applications, including medical imaging, photography, and computer vision.

3. Image restoration

Deconvolution (Wiener Filter): The Wiener filter is defined as:

$$H(u,v) = \frac{S^*(u,v)}{|S(u,v)|^2 + K}$$

where H is the filter, S^* is the complex conjugate of the signal, and K is a constant that represents noise.

Example: If an image is blurred due to motion, applying the Wiener filter can restore the sharpness by reversing the effects of the blur.

4. Edge detection and segmentation of images

4.1. Edge detection

Canny Edge Detector: The Canny method involves several steps, including gradient calculation:

$$G(x,y) = \sqrt{I_x^2 + I_y^2}$$

where I_x and I_y are the gradients in the x and y directions.

Example: Using the Canny edge detector on a photograph will highlight the edges of objects within the image, making them more pronounced.

4.2. Image Segmentation

Thresholding: The basic thresholding operation can be expressed as:

$$I_{segmented}(x,y) = \begin{cases} 1 \; if \; I(x,y) > T \\ 0 \; otherwise \end{cases}$$

where T is the threshold value.

Example: In segmenting a grayscale image of a fruit, setting a threshold can help isolate the fruit from the background based on intensity.

5. Wavelet transform

5.1. Continuous Wavelet Transform (CWT)

$$W(a,b) = \int_{-\infty}^{\infty} f(t)\psi^*(at - b)\, dt$$

where $W(a,b)$ is the wavelet coefficient, $f(t)$ is the signal, ψ is the wavelet function, a is the scale, and b is the translation.

Example: The wavelet transform can be used for image compression by representing an image in terms of its wavelet coefficients, allowing for efficient storage.

6. Other Signal and Image Processing Methods

6.1. Fourier Transform

Discrete Fourier Transform (DFT):

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

where $X(k)$ is the DFT of the signal $x(n)$.

Example: Applying the DFT to an image allows for frequency analysis, helping to identify periodic patterns or noise.

6.2. Morphological Operations

Dilation: The dilation operation can be defined as:

$$(A \oplus B)(x,y) = max_{(i,j)\in B} A(x-i, y-j)$$

where A is the image and B is the structuring element.

Example: Dilation can be used to expand the boundaries of objects in a binary image, useful in shape analysis.

6.3. Principal Component Analysis (PCA)

PCA Transformation:

$$Z = XW$$

where Z is the transformed data, X is the original data, and W is the matrix of eigenvectors.

Example: PCA can reduce the dimensionality of image data while preserving variance, making it easier to analyze high-dimensional images.

7. Clustering and Classification

7.1. Clustering

K-means Clustering: The objective function can be defined as:

$$J = \sum_{i=1}^{k} \sum_{j=1}^{n} \| x_j^{(i)} - \mu_i \|^2$$

where k is the number of clusters, $x_j^{(i)}$ is the data point, and μ_i is the centroid of cluster i.

Example: K-means can be applied to segment an image into different color regions based on pixel intensity.

7.2. Classification

Support Vector Machine (SVM): The decision boundary can be expressed as:

$$f(x) = w^T x + b$$

where w is the weight vector, x is the input feature vector, and b is the bias.

Example: SVM can classify images of handwritten digits by finding the optimal hyperplane that separates different digit classes.

Section 2 : Processing of biomedical signals

1. Electric Activities of the Cell

The electric activities of cells, particularly in neurons and muscle cells, are fundamental for understanding physiological functions and how signals are transmitted within the body. These activities can be measured using various techniques, such as patch-clamp recordings, which allow for the study of ionic currents in individual cells.

1.1. Resting Potential

The resting potential is the voltage difference across the cell membrane when the cell is in a non-active state, typically around $-70\ \text{mV}$$-70\text{mV}$. This potential is maintained by the selective permeability of the cell membrane and the action of ion pumps, primarily the sodium-potassium pump (Na^+/K^+ ATPase), which actively transports potassium ions into the cell and sodium ions out.

1.2. Action Potential

An action potential is a rapid, transient change in membrane potential that occurs when a neuron is stimulated. The Hodgkin-Huxley model describes the dynamics of action potentials in neurons. The equation governing the membrane potential V can be expressed as:

$$C_m \frac{dV}{dt} = I_{in} - I_K - I_{Na} - I_L$$

where:

C_m = membrane capacitance,

V = membrane potential,

I_{in} = input current,

I_K = potassium current,

I_{Na} = sodium current,

I_L = leakage current.

Example: When a neuron is stimulated, sodium channels open, allowing Na^+ ions to flow into the cell, causing depolarization. If the depolarization reaches a threshold, an action potential is generated, propagating along the axon to transmit information.

2. Electrocardiogram (ECG)

An electrocardiogram (ECG) is a recording of the electrical activity of the heart over time, which is crucial for diagnosing various cardiac conditions. It provides insights into heart rhythm, size, and the presence of ischemic damage.

2.1. ECG Waveforms:

The ECG consists of several key components:

P wave: Represents atrial depolarization.

QRS complex: Represents ventricular depolarization.

T wave: Represents ventricular repolarization.

Intervals and Segments:

The PR interval, QT interval, and ST segment are critical for assessing heart function and detecting abnormalities.

The relationship between the heart's electrical activity and the ECG signal can be modeled as:

$$ECG(t) = A \cdot sin(2\pi ft + \phi)$$

where:

A = amplitude (reflecting the strength of electrical activity),

f = frequency (related to heart rate),

ϕ = phase shift (reflecting timing).

Example: An ECG can be used to detect arrhythmias, myocardial infarction, or other heart conditions based on the shape and timing of the P, QRS, and T waves. For instance, a prolonged QT interval may indicate a risk of life-threatening arrhythmias.

3. Electroencephalogram (EEG)

An electroencephalogram (EEG) measures the electrical activity of the brain and is primarily used to diagnose neurological conditions such as epilepsy, sleep disorders, and brain tumors.

3.1. Brain Waves

EEG records different types of brain waves, each associated with different states of consciousness:

Delta waves (0.5-4 Hz): Associated with deep sleep.

Theta waves (4-8 Hz): Associated with light sleep and relaxation.

Alpha waves (8-12 Hz): Associated with a relaxed, calm state, often seen when a person is awake but relaxed.

Beta waves (12-30 Hz): Associated with active thinking, problem-solving, and concentration.

The EEG signal can be represented as:

$$EEG(t) = \sum_{j=1}^{m} A_i \cdot sin(2\pi f_i t + \phi_i)$$

where:

A_i = amplitude of the i^{th} frequency component,

f_i = frequency of the i^{th} component,

ϕ_i = phase shift of the i^{th} component.

Example: EEG is commonly used to diagnose epilepsy by identifying abnormal brain wave patterns during seizures. For instance, a spike-and-wave pattern is characteristic of generalized epilepsy.

4. Electromyogram (EMG)

An electromyogram (EMG) measures the electrical activity of muscles and is used to diagnose neuromuscular disorders, muscle diseases, and nerve damage.

4.1. Motor Unit Action Potential (MUAP):

The electrical signal generated by a motor unit during muscle contraction. EMG can be surface (non-invasive) or intramuscular (invasive) depending on the depth of the measurement.

The EMG signal can be modeled as:

$$EMG(t) = \sum_{j=1}^{m} A_j \cdot sin(2\pi f_j t + \phi_j)$$

where:

A_j= amplitude of the j^{th} muscle fiber,

f_j = frequency of the j^{th} muscle fiber activity,

ϕ_j = phase shift of the j^{th} fiber.

Example: EMG can be used to assess conditions like muscular dystrophy or nerve compression syndromes by analyzing the electrical activity of muscles during contractions. For example, a reduced amplitude of the MUAPs may indicate muscle weakness or neuropathy.

5. Other Biomedical Signals

This category encompasses various other signals used in biomedical applications, including:

5.1. Blood Pressure Waveforms:

Blood pressure is typically measured using sphygmomanometers or arterial catheters. The pressure can be modeled as:

$$P(t) = P_0 + A \cdot e^{\frac{-t}{\tau}} sin(2\pi ft + \phi)$$

where:

P_0 = baseline pressure,

A = amplitude,

τ = time constant (reflecting the damping of the waveform),

f = frequency (related to heart rate).

Example: Continuous blood pressure monitoring is vital in critical care settings to assess patient status and guide treatment.

5.2. Pulse Oximetry:

Pulse oximetry measures the oxygen saturation of hemoglobin in blood. It works by shining red and infrared light through a translucent part of the body (like a fingertip) and measuring the ratio of light absorption, which reflects the oxygen saturation level.

5.3. Respiratory Signals:

Respiratory signals are monitored using spirometry, which can be modeled as:

$$V(t) = V_{max} \cdot sin(2\pi ft + \phi)$$

where:

V_{max} = maximum volume (peak tidal volume),

f = frequency of breathing,

ϕ = phase shift.

Example: Monitoring respiratory rate and volume is essential in assessing lung function and detecting conditions like asthma or chronic obstructive pulmonary disease (COPD)

Section 3 : Processing of biomedical images

General Introduction

As you near the completion of your master's studies in biomedical engineering, particularly in the specialization of Biomedical Imaging Systems, it is essential to grasp the significant impact that medical imaging technologies have on contemporary healthcare. These advanced techniques are fundamental in generating visual representations of the internal structures of the human body. They are pivotal in clinical analysis, medical interventions, and the visualization of organ and tissue functions, thereby fundamentally transforming the ways in which healthcare professionals diagnose and treat diseases.

Medical imaging technologies enable physicians to "see" within the body without the need for invasive surgical procedures. This non-invasive capability is vital for several reasons:

1. Role of medical imaging technologies

Medical imaging technologies empower physicians to "see" inside the body without resorting to invasive surgical procedures. This non-invasive capability is crucial for several reasons:

- **Accurate diagnosis**: by providing detailed images of internal structures, medical imaging allows for precise identification of diseases and conditions. This accuracy is critical in determining the appropriate course of treatment.
- **Effective treatment planning**: with clear visualizations of affected areas, healthcare providers can devise tailored treatment plans that address the specific needs of each patient. This personalized approach enhances the likelihood of successful outcomes.
- **Ongoing disease monitoring**: medical imaging facilitates the continuous assessment of a patient's condition over time. This is particularly important for chronic diseases, where regular monitoring can inform adjustments to treatment strategies.
- **Preventive medicine**: Early diagnosis through imaging technologies can lead to timely interventions, significantly improving patient outcomes. By detecting diseases at an earlier stage, healthcare providers can implement preventive measures that may mitigate the severity of illnesses.

2. Imaging modalities

Understanding the various imaging modalities is essential for your future roles in the biomedical field. Here are some of the most widely used imaging techniques, along with their applications:

- **X-ray**: this traditional imaging modality uses ionizing radiation to create images primarily of bones and certain soft tissues. Healthcare professionals commonly use x-rays to diagnose fractures, infections, and other skeletal conditions.
- **Computed tomography (CT)**: CT scans represent a more advanced form of x-ray technology. They generate cross-sectional images of the body by combining multiple x-ray images processed by a computer. Ct scans are particularly useful for diagnosing complex conditions in the abdomen, chest, and pelvis.

- **Magnetic resonance imaging (MRI)**: MRI utilizes powerful magnets and radio waves to create highly detailed images of soft tissues. This modality is especially effective for visualizing the brain, spinal cord, muscles, and joints, making it invaluable in neurology and orthopedics.
- **Ultrasound**: by employing high-frequency sound waves, ultrasound imaging produces real-time images of soft tissues. It is widely used in obstetrics to monitor fetal development and in cardiology to assess heart function and structure.
- **Nuclear medicine**: techniques such as positron emission tomography (pet) and single photon emission computed tomography (SPECT) involve administering small amounts of radioactive materials to visualize metabolic processes within the body. These modalities are particularly useful for diagnosing cancers, heart diseases, and neurological disorders.
- **Optical imaging**: this innovative technique utilizes light, typically in the near-infrared spectrum, to image tissues. Optical imaging has promising applications in brain and skin imaging, allowing for detailed assessments of tissue health and function.

3. Definition of medical imaging

Medical imaging refers to the techniques and processes used to create visual representations of the interior of the body for clinical analysis and medical intervention. It encompasses a wide range of technologies that visualize the structure and function of organs and tissues, aiding in the diagnosis, treatment, and monitoring of various medical conditions.

4. Importance in medicine

Medical imaging plays a crucial role in modern healthcare by providing critical insights that are essential for effective diagnosis and treatment. It allows healthcare professionals to:

- **Diagnose conditions**: Detect and diagnose a wide array of medical conditions, from fractures and infections to tumors and vascular diseases.
- **Guide interventions**: Assist in planning and guiding surgical and non-surgical interventions with precision, reducing the risk of complications.
- **Monitor disease progression**: Track the progression of diseases and evaluate the effectiveness of treatments, facilitating timely adjustments to therapy.
- **Preventative care**: Enable early detection of diseases, which can lead to more effective and less invasive treatment options.

5. Historical context

The field of medical imaging has evolved significantly since its inception, driven by technological advancements and scientific discoveries:

- **X-rays (1895)**: Wilhelm Conrad Roentgen's discovery of X-rays marked the beginning of medical imaging, allowing the visualization of bones and certain internal organs.
- **Ultrasound (1940s-1950s)**: The development of ultrasound technology provided a non-invasive method for visualizing soft tissues, particularly useful in obstetrics and cardiology.

- **Computed tomography (1970s)**: The advent of CT scanning introduced cross-sectional imaging, enhancing the ability to detect and characterize complex conditions.
- **Magnetic resonance imaging (1980s)**: MRI revolutionized imaging by offering high-resolution images of soft tissues without ionizing radiation, expanding diagnostic capabilities in neurology, orthopedics, and oncology.
- **Nuclear medicine (1950s-present)**: Techniques like PET and SPECT allow healthcare professionals to perform functional imaging that visualizes physiological processes at the molecular level.
- **Recent advances**: Ongoing innovations, such as functional MRI, hybrid imaging techniques (e.g., PET/CT, PET/MRI), and the integration of artificial intelligence, continue to push the boundaries of medical imaging.

6. Key Equations in Medical Imaging

Understanding key equations that govern medical imaging technologies is essential for grasping their principles. Here are some fundamental equations relevant to various imaging modalities:

7.1. X-ray Imaging

The intensity I of an X-ray beam after passing through a material is described by the exponential attenuation law:

$$I = I_0 e^{-\mu x}$$

where:

I_0 = initial intensity of the X-ray beam,

μ = linear attenuation coefficient of the material,

x = thickness of the material.

7.2. Computed tomography (CT)

In CT imaging, the Radon transform is used to reconstruct images from projections:

$$f(x,y) \int_{-\infty}^{\infty} g(s,\theta)\delta(xcos\theta + ysin\theta - s)ds$$

where:

$f(x,y)$ = image function,

$g(s,\theta)$ = projection data,

δ = Dirac delta function.

7.3. Magnetic resonance imaging (MRI)

The signal S in MRI can be described by the equation:

$$S = M_0 . e^{-t/T_2} . (1 - e^{-t/T_1})$$

where:

M_0 = equilibrium magnetization,

T_1 = longitudinal relaxation time,

T_2 = transverse relaxation time,

t = time after the RF pulse.

7.4. Ultrasound imaging

The speed of sound c in a medium is given by:

$c=\lambda f$

where:

λ = wavelength,

f = frequency.

7.5. Nuclear medicine

In PET imaging, the relationship between the detected counts C and the activity A of the radioactive tracer is:

$C=\epsilon A$

where:

ϵ = detection efficiency.

7. Conclusion

As you delve deeper into these imaging modalities in the **biomedical imaging system** course, consider not only their technological foundations but also their clinical applications and the ongoing innovations that continuously shape the field of biomedical engineering. Understanding these technologies and their underlying equations will empower you to contribute meaningfully to advancements in healthcare, ultimately improving patient care and outcomes. Embrace this opportunity to explore the fascinating world of medical imaging, where science and technology converge to enhance lives.

Chapter 1: X-ray

1. Introduction

X-ray imaging serves as an essential area of study for biomedical engineering students, presenting a unique intersection of engineering principles, medical applications, and cutting-edge technology. By exploring the mechanics behind X-ray imaging, students gain valuable insights into how professionals design, optimize, and utilize these systems in clinical settings. This hands-on understanding empowers students to engage actively with the technology and its applications, preparing them for impactful careers in healthcare.As future engineers, students can explore the development of advanced imaging technologies, such as digital radiography and computed tomography (CT), which leverage sophisticated algorithms and software to enhance image quality and diagnostic capabilities.

Moreover, the role of biomedical engineers in improving patient safety and imaging efficiency is critical. Students can engage in research and projects focused on minimizing radiation exposure while maximizing diagnostic accuracy. This involves not only understanding the physics of X-ray generation and detection but also exploring innovative solutions, such as image processing techniques and machine learning algorithms that can aid in automating and refining image analysis. By delving into these topics, biomedical engineering students can contribute to the ongoing evolution of medical imaging technologies, ultimately improving patient care and outcomes.

Additionally, the interdisciplinary nature of biomedical engineering allows students to collaborate with healthcare professionals, radiologists, and physicists, fostering a comprehensive understanding of the clinical implications of imaging technologies. This collaboration can lead to the development of novel imaging modalities and techniques that address specific medical challenges. By engaging with real world, applications and challenges in X-ray imaging, biomedical engineering students can cultivate a passion for innovation and problem solving, positioning themselves as key contributors to the future of healthcare technology.

2. Basic physics of X-rays

2.1. Generalities

X-rays are a form of **electromagnetic radiation** with much higher energy than visible light, capable of ionizing atoms and molecules. This ionization can lead to chemical changes in biological tissues, which is the basis for their use in medical imaging and treatments.

2.2. Properties

- **Wavelength Range**: X-rays have wavelengths typically ranging from **0.01 to 10nm**.
- **Frequency Range**: This corresponds to frequencies of approximately 3×10^{15} Hz(PetaHz) to 3×10^{30}Hz (ExaHz).

- The relationship between wavelength (λ) and frequency (f) can be described by the equation:

$$c = \lambda f$$

Where

c is the speed of light in a vacuum ($C \approx 3 \times 10^8$ m/s).

2.3. Production of X-Rays

X-ray tubes are the primary components of X-ray machines. The basic structure of an X-ray tube includes:

- **Cathode**: The negative electrode that emits electrons when heated.
- **Anode**: The positive electrode where the electrons collide.
- **Vacuum Enclosure**: Prevents the electrons from colliding with air molecules.

See figure 1.

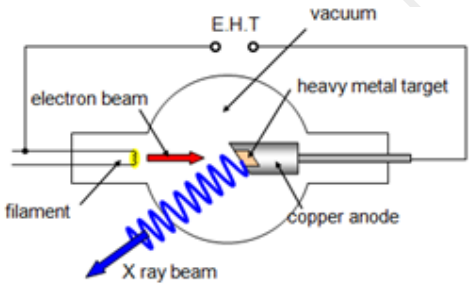


Figure 1: Production of X-Rays

2.4. Mechanism of X-Ray production

Thermionic emission: is the process by which electrons are emitted from a material (usually a metal) when it is heated to a high temperature. As the temperature of the cathode increases, the thermal energy provided to the electrons allows some of them to overcome the attractive forces binding them to the material. This results in the emission of electrons into the surrounding vacuum or gas. When the cathode is heated, it emits electrons due to thermionic emission. The number of emitted electrons can be described by the Richardson equation:

$$I = AT^2 e^{-kT\phi}$$

where:

- I is the current (number of emitted electrons),
- A is the Richardson constant,
- T is the absolute temperature of the cathode in Kelvin,

- ϕ is the work function of the material, which is the minimum energy needed to remove an electron from the surface of the material. A higher work function means that more energy is required for an electron to escape.
- k is the Boltzmann constant, $K = 1.38 \times 10^{-23} J/K$

Thermionic emission is a critical phenomenon in various applications, such as vacuum tubes and cathode ray tubes. The Richardson equation quantitatively describes how the emission of electrons from a heated cathode depends on temperature and material properties, allowing for predictions about the behavior of thermionic devices.

Acceleration: A high voltage potential (typically in the range of 30-150 kV) accelerates these electrons toward the anode.

Collision and X-ray generation: When the high-speed electrons collide with the anode material (usually tungsten), they undergo rapid deceleration. This process generates X-ray photons through two main mechanisms:

- **Bremsstrahlung radiation**: This occurs when electrons are deflected by the electric field of the nuclei in the anode material, resulting in the emission of X-ray photons. The energy of the emitted photons can be calculated using:

$$E = eV$$

Where E is the energy of the X-ray photon, e is the charge of the electron, and V is the accelerating voltage.

- **Characteristic Radiation**: This occurs when an incoming electron knocks out an inner-shell electron from the anode material, causing an outer-shell electron to fall into the vacancy, emitting an X-ray photon with energy characteristic of the anode material. The energy of the characteristic X-ray can be expressed as:

$$E = E_{shell1} - E_{shell2}$$

where E_{shell1} is the energy of the higher energy shell and E_{shell2} is the energy of the lower energy shell.

2.5. Interaction of X-Rays with matter

When X-rays pass through matter, they can interact in several ways:

Photoelectric Effect: An X-ray photon is completely absorbed by an atom, resulting in the ejection of an inner-shell electron. This effect is significant at lower energies and for high atomic number materials.

Compton Scattering: An X-ray photon collides with a loosely bound outer-shell electron, resulting in partial energy transfer. The photon is scattered at a lower energy and different angle. The energy and angle can be described by:

$$E' = \frac{E}{1 + \frac{E}{m_e c^2 (1 - cos\theta)}}$$

Where E is the energy of the incoming photon, E' is the energy of the scattered photon, m_e is the mass of the electron, c is the speed of light, and θ is the scattering angle.

Rayleigh Scattering: This elastic scattering occurs without energy loss, primarily at low energies and in small particles.

Pair Production: At very high energies (greater than 1.022 MeV), an X-ray photon can produce an electron-positron pair when interacting with a nucleus.

X-rays play a crucial role in medical imaging and treatment due to their unique properties and interactions with matter. Understanding their production and behavior is essential for optimizing their use in various applications, including diagnostic radiology and radiation therapy.

3. Absorption and attenuation of X-Rays:

3.1. Photoelectric Effect

The **photoelectric effect** occurs when an X-ray photon is completely absorbed by an atom, resulting in the ejection of an inner-shell electron. This process is critical in medical imaging, as it enhances the contrast of X-ray images, particularly in tissues with different atomic numbers. Lets discus the key characteristics in summary as bellow

- **Energy Dependence**: The probability of the photoelectric effect occurring is highly dependent on the energy of the X-ray photon (E) and the atomic number (Z) of the absorbing material. The photoelectric absorption coefficient (μ_{PE}) can be approximated by the following relationship:

$$\mu_{PE} \propto \frac{Z^3}{E^3}$$

This indicates that higher atomic number materials (like lead) are more effective at absorbing X-rays, enhancing image contrast.

- **Threshold Energy**: The minimum energy required to eject an electron from an inner shell is given by the work function (ϕ) of the material. For the photoelectric effect to occur, the energy of the incident photon must satisfy:

$$E \geq \phi$$

Where E is the energy of the incoming X-ray photon.

3.2. Contrast enhancement

The increased absorption in high-Z materials leads to greater differences in attenuation between different tissues, which is crucial for producing clear images. The contrast (C) in an X-ray image can be quantified as:

$$C = \frac{I_1 - I_2}{I_1 + I_2}$$

where I_1 and I_2 are the intensities of X-rays transmitted through different tissues.

See figure 2:

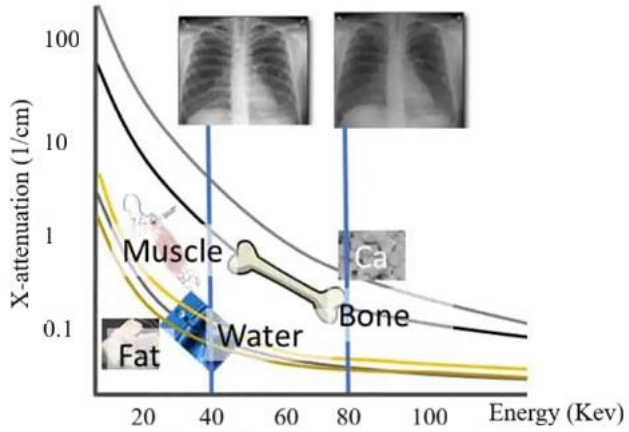


Figure 2: Contrast in X-ray images

3.3. Compton scattering

Compton scattering occurs when X-ray photons interact with loosely bound outer-shell electrons, resulting in a reduction in energy (and thus a longer wavelength) and a change in direction. This scattering contributes to image noise and reduces overall image contrast. See figure 3.

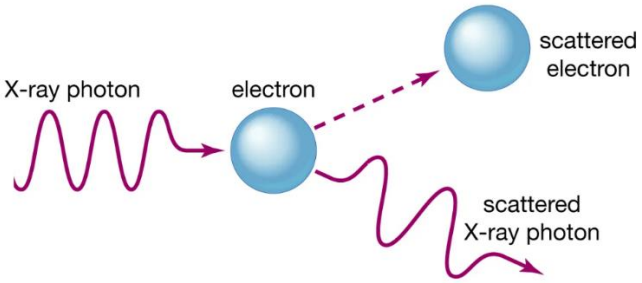


Figure 3: Compton scattering

- **Energy and wavelength change**: The energy of the scattered photon can be described by the Compton scattering formula:

$$E' = \frac{E}{1 + \frac{E}{m_e c^2 (1 - cos\theta)}}$$

Where:

E is the energy of the incident photon,

E' is the energy of the scattered photon,

m_e is the mass of the electron (9.11×10^{-31}kg),

c is the speed of light (3×10^8 m/s),

θ is the scattering angle.

- **Change in wavelength**: The change in wavelength ($\Delta\lambda$) due to Compton scattering is given by:

$$\Delta\lambda=\lambda'-\lambda=\frac{h}{m_e c}(1-cos\theta)$$

Where:

λ is the initial wavelength,

λ' is the wavelength after scattering,

h is Planck's constant (6.626×10^{-34} J s).

- **Impact on Image Quality:** The scattered photons can contribute to image noise, as they can scatter in various directions, reducing the clarity of the image. The overall attenuation μ_c due to Compton scattering can be described by:

$$\mu_c=\mu_{C0}.\rho.\frac{E_0}{E}$$

Where:

μ_{C0} is the mass attenuation coefficient for Compton scattering,

ρ is the density of the material,

E_0 is the initial energy of the photon,

E is the energy after scattering.

Both the photoelectric effect and Compton scattering play significant roles in the absorption and attenuation of X-rays in various materials. Understanding these processes and their associated equations is crucial for optimizing X-ray imaging techniques and enhancing image quality.

4. Transmission:

4.1. Penetration of X-Rays

X-rays have the ability to penetrate various materials, including human tissues. The extent of penetration is influenced by several factors, primarily the energy of the X-rays, as well as the density and composition of the material they encounter. Factors Affecting Penetration

- **Energy of X-Rays**: Higher energy X-rays have greater penetrating power. The energy of an X-ray photon is given by:

$$E = hf$$

Where:

E is the energy of the photon,

h is Planck's constant (6.626×10^{-34} Js),

f is the frequency of the X-ray.

As the frequency increases (and thus the wavelength decreases), the energy of the X-ray photons increases, allowing them to penetrate more dense materials.

Material density: The density (ρ) of the material also plays a crucial role in determining how much X-ray radiation is absorbed or transmitted. Denser materials have more atoms per unit volume, which increases the likelihood of interactions with X-ray photons.

Composition of material: The atomic number (Z) of the elements in the material significantly affects X-ray penetration. Materials with higher atomic numbers (like lead) are more effective at absorbing X-rays due to increased photoelectric absorption and Compton scattering.

4.2. Attenuation and half-value layer

The **attenuation** of X-rays as they pass through a material can be quantitatively described by the exponential attenuation law:

$$I = I_0 e^{-\mu x}$$

Where:

I is the intensity of the X-ray after passing through a distance x,

I_0 is the initial intensity of the X-ray,

μ is the linear attenuation coefficient of the material, which depends on both the energy of the X-rays and the material properties.

The **linear attenuation coefficient** (μ) can be expressed as:

$\mu=\mu_{PE}+\mu C+\mu_R$

Where:

μ_{PE} is the contribution from the photoelectric effect,

μ_C is the contribution from Compton scattering,

μ_R is the contribution from Rayleigh scattering.

The **half-value layer (HVL)** is defined as the thickness of a material required to reduce the intensity of X-rays to half its original value. It is calculated from the linear attenuation coefficient:

$$HVL = \frac{ln(2)}{\mu}$$

Where $\ln(2) \approx 0.693$. The HVL provides a useful measure of the penetrating ability of X-rays through different materials.

The penetration of X-rays through various materials, including human tissues, is a critical aspect of their use in medical imaging. Understanding the factors that influence this penetration, such as X-ray energy, material density, and composition, along with the relevant equations, is essential for optimizing imaging techniques and ensuring effective diagnostic outcomes.

5. Contrast in imaging

5.1. Differential absorption

Differential absorption refers to the varying degrees to which different tissues in the body absorb X-rays. This phenomenon is crucial for producing contrast in radiographic images, allowing healthcare professionals to distinguish between different types of tissues based on their absorption characteristics. Mechanisms of Differential Absorption

Density of Tissues: The density (ρ) of tissues plays a significant role in how much X-ray radiation is absorbed. Denser materials contain more mass per unit volume, leading to greater interaction with X-ray photons.

Atomic Number: The atomic number (Z) of the elements in the tissue also affects absorption. Higher atomic number elements have a greater probability of interacting with X-ray photons, primarily through the photoelectric effect. For example, bones contain calcium (atomic number 20), which is denser and has a higher atomic number compared to the elements predominantly found in soft tissues (like carbon, oxygen, and hydrogen).

5.2. Absorption coefficients

The **linear attenuation coefficient** (μ) quantifies how much X-ray intensity decreases as it passes through a material. It is influenced by both the energy of the X-rays and the composition of the material. The relationship can be expressed as we described above.

6. Mathematical description

6.1. Beer-Lambert law

The **Beer-Lambert law** is a fundamental principle that describes how the intensity of light (or other electromagnetic radiation, including X-rays) decreases as it passes through a medium. This law is particularly useful in various fields, including physics, chemistry, and medical imaging, to quantify the attenuation of X-rays through different materials. The Beer-Lambert law can be mathematically expressed as described above. However, we should emphasise on: components of the equation.

Transmitted intensity (I): This is the amount of X-ray intensity that emerges from the material after interacting with it. It reflects how much of the incident X-ray beam has been absorbed or scattered.

Initial intensity (I_0): This represents the intensity of the X-ray beam before it interacts with the material. It is the reference point for measuring attenuation.

Linear attenuation coefficient (μ): This coefficient quantifies how easily a material can attenuate X-rays. It is influenced by:

The material's density (ρ): Denser materials generally have higher attenuation coefficients.

The atomic number (Z): Materials with higher atomic numbers tend to absorb X-rays more effectively due to increased interactions (photoelectric effect and Compton scattering).

The energy of the X-rays: Higher energy X-rays may penetrate materials more easily, resulting in lower attenuation coefficients.

Thickness of Material (x): The distance the X-rays travel through the material directly affects the extent of attenuation. Thicker materials will result in greater absorption and scattering.

See figure 4, the attenuation law figure.

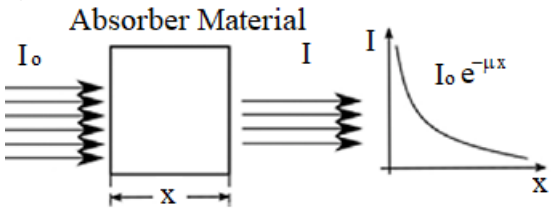


Figure 4: Attenuation law

6.2. Applications in medical imaging

In medical imaging, the Beer-Lambert Law helps in understanding how different tissues absorb X-rays. For instance:

- **Bone vs. soft tissue**: Bones, being denser and containing higher atomic number elements (like calcium), have a higher linear attenuation coefficient compared to soft tissues,

which are less dense and primarily consist of lighter elements (like carbon and oxygen). This difference in attenuation leads to the high contrast observed in radiographs, where bones appear white (high absorption) and soft tissues appear in shades of gray (lower absorption).

- **Quantitative analysis**: The Beer-Lambert Law allows radiologists to estimate the thickness and density of tissues based on the intensity of X-rays that are transmitted. By analyzing the intensity of the X-ray beam before and after it passes through a patient, medical professionals can infer important diagnostic information.

6.3. Limitations

While the Beer-Lambert Law is widely applicable, it has some limitations, especially in complex biological tissues where multiple scattering events and non-linear effects may occur. Additionally, it assumes a uniform medium and does not account for variations in tissue composition or structure.

The Beer-Lambert Law is a crucial tool in understanding the attenuation of X-rays as they pass through various materials, particularly in medical imaging. By quantifying how different tissues absorb X-rays, this law enables the production of clear and informative diagnostic images, facilitating accurate medical assessments.

7. Image formation:

7.1. Projection imaging

In **conventional radiography**, a **2D image** is formed by projecting a **3D structure** onto a detector. This process involves the following key concepts: see figure 5.

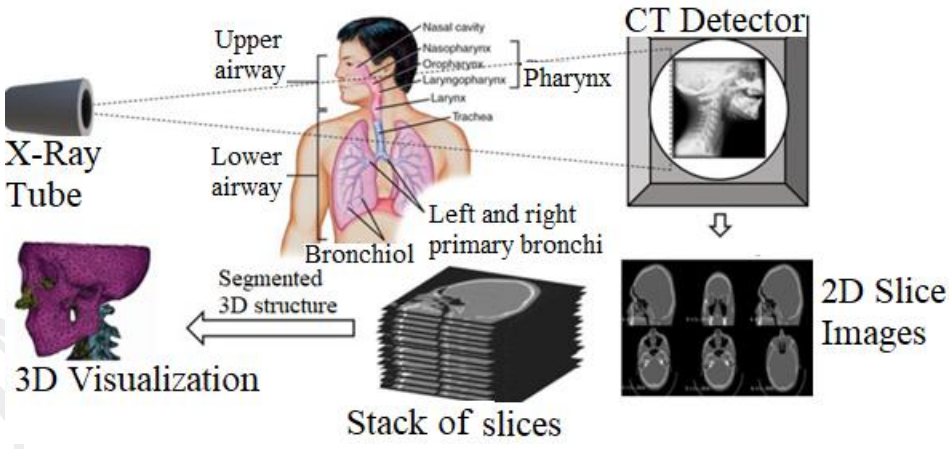


Figure 5: Image formation

- **X-ray generation**: X-rays are produced by the interaction of high-energy electrons with a target material (usually tungsten) in the X-ray tube. The energy of the emitted X-ray photons is typically in the range of 20-150 keV.
- **Path of X-ray beam**: As X-rays pass through the body, they interact with various tissues, undergoing **absorption** and **scattering**. The intensity of the X-rays reaching the detector depends on the cumulative absorption along the path of the X-ray beam.

- **Cumulative absorption**: The intensity of the X-rays that reach the detector can be described using the Beer-Lambert Law.

7.2. Image formation process

The formation of a radiographic image involves the following steps:

- **X-ray emission**: X-rays are emitted from the tube and directed towards the patient.
- **Tissue interaction**: X-rays pass through various tissues, with different degrees of absorption based on the tissue density and atomic number.
- **Detection**: The remaining X-rays that pass through the body strike a detector (film or digital), forming a latent image.
- **Image processing**: The latent image is processed to produce a visible image, highlighting the differences in absorption between various tissues.

8. Applications

8.1. Modern digital detectors

- **Digital conversion**: Modern digital detector systems convert X-ray photons directly into digital signals. This conversion allows for enhanced image processing, storage, and retrieval, significantly improving diagnostic capabilities.
- **Reduced radiation dose**: Digital systems often require lower doses of radiation compared to traditional film-based systems. This is due to their higher sensitivity and the ability to manipulate image contrast and brightness digitally. See figure 6.

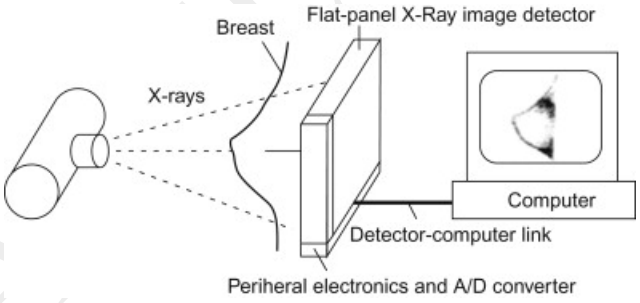


Figure 6: Modern digital detectors

8.2. Specialized techniques

- **CT Scanning (computed tomography)**:
- **Mechanism**: CT scans use X-rays taken from multiple angles around the body to create detailed cross-sectional images. The mathematical reconstruction of these images is typically done using algorithms like the **Filtered back projection** or **iterative reconstruction**.
- **Mathematical representation**: The image reconstruction can be expressed in terms of the Radon transform, which relates the projection data to the original image function.
- **Fluoroscopy**:

- **Real-time imaging**: This technique provides real-time moving images of the internal structures of the body, making it useful for guiding procedures such as catheter insertions.
- **Continuous X-ray exposure**: Fluoroscopy involves continuous exposure to X-rays, necessitating careful management of radiation dose.
- **Mammography**:
- **Specialized imaging**: Mammography is a specialized form of X-ray imaging designed specifically for breast tissue. It uses lower radiation doses and higher contrast to detect abnormalities.
- **Screening and diagnosis**: It is crucial for early detection of breast cancer, often employing techniques like digital mammography or tomosynthesis.

8.3. Safety and radiation protection

Understanding the theory of X-rays is fundamental for appreciating how this technology works and its wide range of applications in medical diagnostics. The physics of X-ray production, interaction with matter, and the principles of image formation are critical for effectively and safely using X-ray imaging in healthcare.

8.3.1. Radiation dose management

Healthcare professionals must carefully manage radiation doses because X-rays are a form of ionizing radiation that can damage living tissues. To minimize exposure and avoid adverse effects such as:

- **Radiation burns**
- **Radiation sickness**
- **Increased cancer risk**

8.3.2. Protection Measures

To safeguard patients and healthcare workers from unnecessary exposure, several protection measures should be respected:

- **Thyroid shields**: Protect the thyroid gland from scatter radiation during X-ray procedures.
- **Lead aprons**: Use lead aprons to shield other parts of the body.
- **Distance and shielding**: Maintain distance from the radiation source and use barriers to reduce exposure.

In conclusion, the principles of image formation in X-ray imaging, including projection imaging and modern digital detector systems, are essential for effective medical diagnostics. Understanding these principles, along with the importance of radiation safety and protection measures, ensures the safe and effective use of X-ray technology in healthcare.

Understanding the theory of X-rays is fundamental for appreciating how this technology works and its wide range of applications in medical diagnostics. The physics of X-ray production, interaction with matter, and the principles of image formation are critical for effectively and safely using X-ray imaging in healthcare. Healthcare professionals must use a specific radiation dose because X-rays are ionizing radiation that can damage living tissues. They must minimize exposure to avoid adverse effects such as radiation burns, radiation sickness, and an increased risk of cancer. We must respect protection measures, such as using thyroid shields and other protective barriers, to safeguard patients and healthcare workers from unnecessary exposure.

Chapter 2. Magnetic resonance imaging (MRI)

1. Introduction

Magnetic Resonance Imaging (MRI) is a powerful non-invasive imaging technique widely used in medical diagnostics. It leverages the principles of nuclear magnetic resonance (NMR) to produce detailed images of the organs and tissues within the body. MRI is particularly valuable due to its ability to generate high-resolution images without the use of ionizing radiation, making it a safer alternative to X-rays and CT scans.

MRI aligns hydrogen nuclei (protons) in the body using a strong magnetic field. When radiofrequency (RF) pulses apply, these protons temporarily shift out of alignment. As they return to their original state, they emit signals that sophisticated algorithms detect and convert into images. Various factors, including tissue composition, water content, and the presence of specific metabolites, influence the contrast in MRI images. Medical professionals use MRI across a range of specialties, including:

Neurology: For imaging brain structures and diagnosing conditions such as tumors, strokes, and neurodegenerative diseases.

Orthopedics: To assess joint injuries, cartilage damage, and soft tissue abnormalities.

Cardiology: For evaluating heart structures and function, as well as detecting cardiac diseases.

Oncology: To identify and monitor tumors in various body regions.

Recent advancements in MRI technology have led to improved image quality, faster scanning times, and enhanced patient comfort. Techniques such as functional MRI (fMRI) allow for the visualization of brain activity by measuring changes in blood flow, while diffusion-weighted imaging (DWI) provides insights into tissue integrity.

For biomedical imaging system engineers, understanding the intricacies of MRI technology is crucial for the development and optimization of imaging systems. As the field continues to evolve with innovations in hardware and software, engineers play a vital role in enhancing the capabilities of MRI, ultimately improving diagnostic accuracy and patient outcomes.

2. Basic physics

Magnetic Resonance Imaging (MRI) is a sophisticated medical imaging technique that leverages powerful magnets, radio waves, and advanced computing to generate detailed images of internal organs and tissues. Unlike X-ray and CT scans, MRI operates without ionizing radiation, making it a safer option for various diagnostic applications. The underlying physics of MRI is rooted in the principles of nuclear magnetic resonance (NMR). See figure 1.

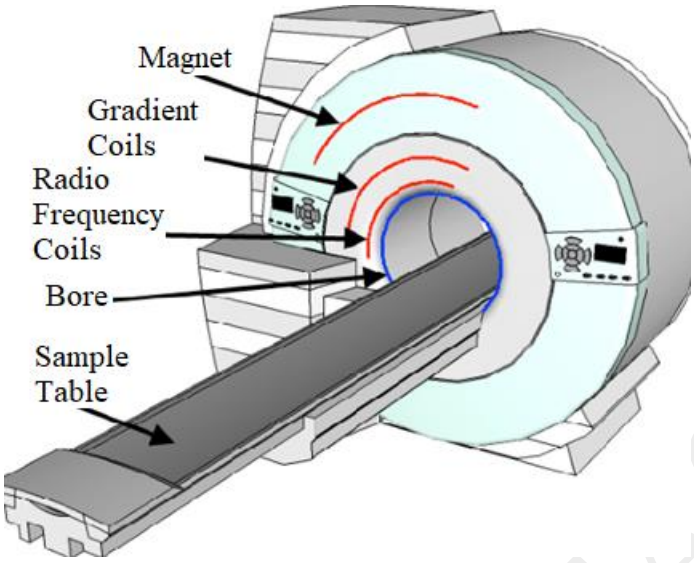


Figure 1: Magnetic resonance imaging (MRI) as a medical imaging

2.1. Fundamental concepts

2.1.1. Nuclear magnetic resonance (NMR)

Certain atomic nuclei, as hydrogen-1 (^1_1H) found in water and fat, possess an intrinsic property known as spin. This spin generates a magnetic moment, causing these nuclei to behave like tiny magnets. See figure 2.

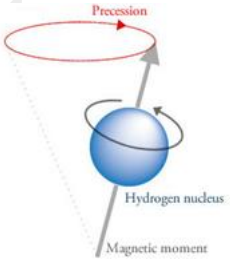


Figure 2: Hydrogen magnetic field

2.1.2. Alignment in a magnetic field

When these nuclei are subjected to a strong external magnetic field field ($\vec{B}_0$), their magnetic moments align either parallel or anti-parallel to the field. The parallel alignment, which is slightly lower in energy, leads to a majority of nuclei aligning in this manner, resulting in a net magnetization along the direction of the magnetic field. See figure 3.

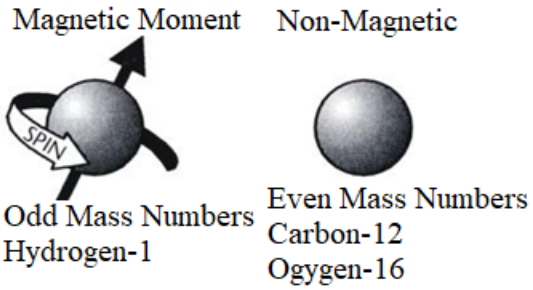


Figure 3: Magnetic proprieties for nuclei

2.1.3. Larmor frequency

Precession:

The magnetic moments of the nuclei do not align perfectly with the external magnetic field; instead, they precess around the direction of the field. This precession occurs at a specific frequency, known as the Larmor frequency, which is determined by the strength of the magnetic field and the type of nucleus. See figure 3.

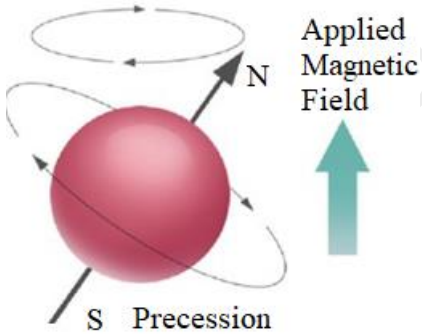


Figure 4: Larmor precession

Equation:

The Larmor frequency (ω_0) is given by:

$\omega_0 = \gamma B_0$

Where:

ω_0 is the Larmor frequency.

γ is the gyromagnetic ratio (a constant specific to each type of nucleus).

B_0 is the strength of the external magnetic field.

2.2. Radiofrequency (RF) pulses

2.2.1. Resonance

When a radiofrequency pulse is applied at the Larmor frequency, energy is transferred to the nuclei, resulting in a tipping of the net magnetization away from the direction of the magnetic field. $\vec{B}_0$. See figure 5.

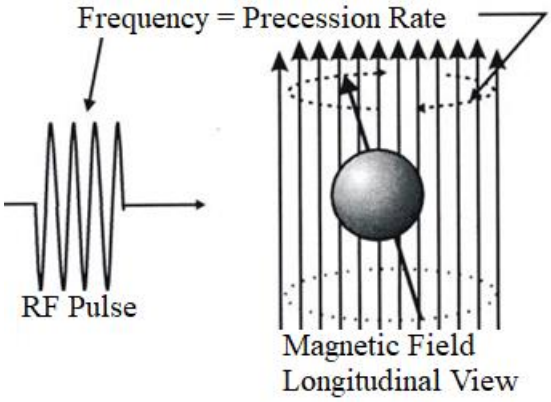


Figure 5: Radiofrequency pulse

2.2.3. Flip angle

The angle by which the net magnetization is tipped is referred to as the flip angle. Commonly used flip angles are 90° (for maximum signal) and 180° (for inversion recovery techniques).

MRI is a powerful imaging modality that relies on the principles of NMR, utilizing the unique properties of atomic nuclei in a magnetic field. By understanding the fundamental concepts of nuclear magnetic moments, Larmor frequency, RF pulses, relaxation processes, and signal detection, one can appreciate the intricate workings of MRI technology in clinical practice.

3. Signal generation in MRI

3.1. Relaxation processes

In MRI, relaxation processes are crucial for signal generation and image formation. After the application of a radiofrequency (RF) pulse, the behavior of the net magnetization vector is characterized by two primary relaxation processes: T_1 and T_2 relaxation.

3.1.1. T_1 Relaxation (Longitudinal relaxation)

Definition

T1 relaxation, commonly referred to as spin-lattice relaxation, involves the process where the net magnetization vector returns to its equilibrium position, aligning with the external magnetic field $\vec{B}_0$ after turning off the radiofrequency (RF) pulse.

During the RF pulse, the applied energy excites the protons, causing them to move away from their equilibrium state. Once the RF pulse ceases, the protons begin to lose this excess energy to their surrounding environment, or lattice. This energy exchange facilitates the realignment of the net magnetization vector with the external magnetic field.

The rate at which this relaxation occurs depends on various factors, including the type of tissue and its molecular environment. Different tissues exhibit distinct T1 relaxation times, which contribute to the contrast observed in MRI images. For instance, fat typically shows a shorter T1 relaxation time compared to water, leading to differences in signal intensity and image

characteristics. Understanding T1 relaxation is crucial for optimizing MRI protocols and enhancing image quality. See figure 6.

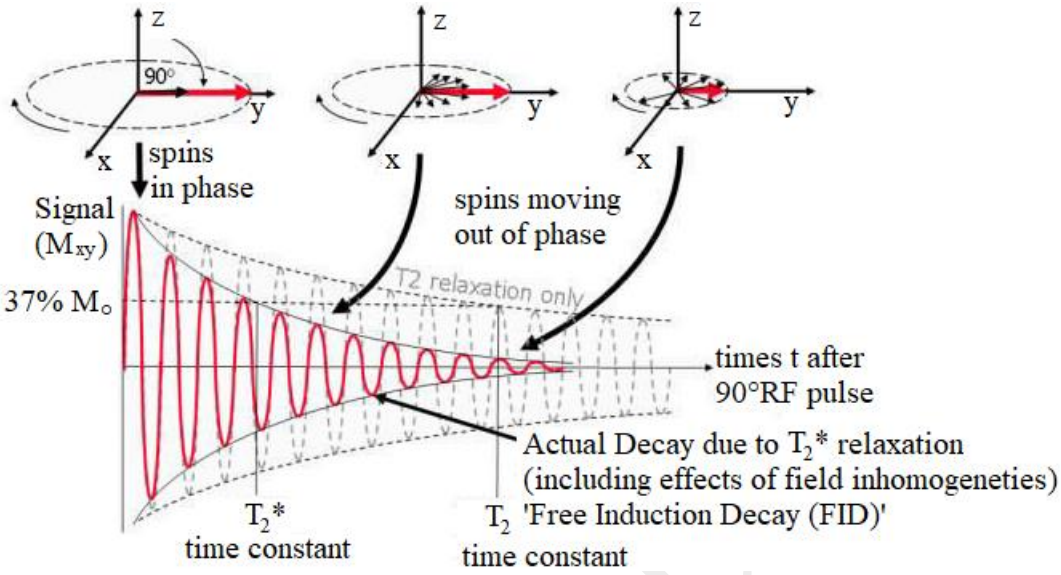


Figure 6: T_1 relaxation

Mechanism

During T_1 relaxation, energy is exchanged between the excited nuclear spins and their surrounding lattice (the molecular environment), allowing the spins to return to their low-energy state.

Time constant

The time it takes for approximately 63% of the net magnetization to recover to its equilibrium state is defined as the T_1 time constant. The recovery can be described mathematically by the equation:

$$M_Z(t) = M_0(1 - e^{-t/T1})$$

Where:

$M_z(t)$ is the longitudinal magnetization at time t.

M_0 is the equilibrium magnetization.

T_1 is the time constant for T_1 relaxation.

T_1 Values: T_1 values vary significantly between different tissues, influencing the contrast in MRI images. For example, T1 is typically longer in fatty tissues than in water-rich tissues.

3.1.2. T2 Relaxation (transverse relaxation)

Definition:

T2 relaxation, often called spin-spin relaxation, describes the process where the transverse component of the magnetization vector decays over time due to interactions among neighboring spins. See figure 6.

When protons in a magnetic field are excited by a radiofrequency (RF) pulse, they become misaligned relative to the magnetic field. This excitation creates a net magnetization vector that has both longitudinal (along the magnetic field) and transverse (perpendicular to the magnetic field) components. While T1 relaxation focuses on the recovery of the longitudinal component, T2 relaxation specifically addresses the loss of coherence among the spins in the transverse plane.

As the excited protons start to interact with each other, these interactions lead to variations in the local magnetic fields experienced by each proton. These magnetic field fluctuations cause the spins to dephase, meaning they lose their synchronized orientation. This dephasing results in a gradual reduction of the transverse magnetization, which manifests as a decay in the signal measured during an MRI scan.

The rate of T2 relaxation varies among different tissues due to their unique molecular environments and the presence of various factors, such as water content and the structure of the tissue. For example, tissues with high water content, like cerebrospinal fluid, typically exhibit longer T2 relaxation times, while denser tissues, such as muscle or fibrous tissue, show shorter T2 times. This variation in T2 relaxation times contributes significantly to the contrast seen in MRI images, allowing radiologists to differentiate between various types of tissues and identify abnormalities. Understanding T2 relaxation is essential for optimizing imaging techniques and improving diagnostic capabilities in MRI.

Mechanism: As the spins in the transverse plane interact with each other, they lose coherence, leading to a reduction in the net transverse magnetization. This process is influenced by factors such as magnetic field inhomogeneities and spin interactions.

Time Constant: The time constant for T_2 relaxation indicates how quickly the transverse magnetization decays. The mathematical representation of this decay is given by:

$$Mxy(t) = Mxy(0)e^{-t/T2}$$

Where:

$M_{xy}(t)$ is the transverse magnetization at time t.

$M_{xy}(0)$ is the initial transverse magnetization.

T_2 is the time constant for T2 relaxation.

T2 Values: Similar to T1, T2 values also vary by tissue type, affecting the contrast and quality of the MRI images. T2 values are generally shorter in denser tissues.

3.1.3. Free Induction Decay (FID)

Signal Detection: After the RF pulse, the transverse magnetization induces a voltage signal in the receiver coil, known as the free induction decay (FID) signal. This signal represents the decaying transverse magnetization and contains essential information about the tissue being imaged. See figure 6.

Mathematical Representation: The FID signal can be expressed as a function of time, taking into account both T1 and T2 relaxation effects:

$$Mxy(0).e^{-t/T2}.e^{-t/T1}$$

Where:

$FID(t)$ is the signal detected at time t.

The first exponential term represents the transverse decay (T2), while the second term accounts for the longitudinal recovery (T1).

Digitization: The FID signal is detected and digitized by the MRI system, allowing for further processing, such as Fourier transformation, to reconstruct the final image.

Understanding the relaxation processes of T1 and T2, along with the generation of the FID signal, is fundamental to MRI technology. These processes not only influence the quality and contrast of the images produced but also provide critical insights into the physiological and pathological conditions of tissues. By manipulating these relaxation times through various imaging sequences, radiologists can optimize MRI protocols for specific diagnostic needs.

4. Image formation in MRI

The process of image formation in Magnetic Resonance Imaging (MRI) is a sophisticated and intricate sequence of steps that ensures high-quality imaging of the body's internal structures. This process begins with spatial encoding, which defines the location of signals within the imaging volume. Following this, slice selection allows for the isolation of specific anatomical sections, enabling detailed examination of targeted areas.

Next, frequency and phase encoding techniques are employed to enhance the spatial resolution and contrast of the images, capturing the nuances of tissue characteristics. Finally, the raw data is transformed into a comprehensible image through image reconstruction using the Fourier transform, a mathematical technique that converts frequency data into spatial information.

Each of these components is essential, working in harmony to provide accurate and detailed visualizations. This intricate process not only aids in diagnosing medical conditions but also enhances our understanding of human anatomy and physiology, making MRI a vital tool in modern medicine.

4.1. Spatial encoding

Spatial encoding in Magnetic Resonance Imaging (MRI) is a critical process that allows for the localization of signals within a three-dimensional space. This technique is essential for creating detailed images of internal structures in the body. Understanding spatial encoding is crucial for engineers involved in MRI technology development. It informs the design of MRI systems, optimization of imaging sequences, and enhancements in image quality. Knowledge of these principles also aids in troubleshooting and improving existing MRI protocols. By mastering spatial encoding, engineers can contribute to advancements in MRI technology, leading to better diagnostic capabilities and patient outcomes.

4.1.1. Gradient magnetic fields

Definition: Gradient magnetic fields are additional magnetic fields applied in specific directions (x, y, and z axes) during an MRI scan. These gradients modify the main magnetic field $\overrightarrow{B_0}$ to create a spatially varying magnetic field.

Effect on Larmor Frequency: The introduction of gradient fields causes the Larmor frequency of the nuclei to vary linearly with their position. This relationship can be expressed mathematically as:

$$\omega(x)=\gamma(B_0+G_x \cdot x)$$

Where:

$\omega(x)$ is the Larmor frequency at position x.

γ is the gyromagnetic ratio of the nucleus.

B_0 is the strength of the main magnetic field.

G_x is the gradient strength along the x-axis.

x is the position along the x-axis.

This variation allows for the differentiation of signals from different locations within the body, enabling the spatial encoding of the signals.

4.2. Slice Selection

4.2.1. Selective RF Pulses

Mechanism: To obtain images from a specific slice of tissue, a gradient field is applied along one axis (e.g., the z-axis), and an RF pulse is transmitted at a frequency that corresponds to the Larmor frequency of the nuclei in that slice.

Slice Thickness: The thickness of the selected slice can be controlled by the bandwidth of the RF pulse. A narrower bandwidth results in a thinner slice. The relationship can be expressed as:

$$\Delta f = \frac{\gamma \cdot Gz \cdot \Delta z}{2\pi}$$

Where:

Δf is the bandwidth of the RF pulse.

G_z is the gradient strength along the z-axis.

Δz is the thickness of the slice.

By varying the gradient and RF frequency, different slices can be selectively excited.

4.3. Frequency and Phase Encoding

4.3.1. Frequency encoding

Mechanism: During signal acquisition, a gradient field is applied along one axis (e.g., the x-axis). This causes the nuclei in different positions to precess at different frequencies, effectively encoding spatial information along that dimension.

Frequency Variation: The frequency of the precessing nuclei can be expressed as:

$\omega_f = \gamma(B_0 + G_x \cdot x)$

Where:

ω_f is the frequency of precession for nuclei at position x.

4.3.2. Phase encoding

Mechanism: Before signal acquisition, a brief gradient field is applied along a perpendicular axis (e.g., the y-axis). This causes the nuclei to precess at different phases based on their position, encoding spatial information along the other dimension.

Phase Shift: The phase shift ϕ introduced by the gradient can be described as:

$\phi = \gamma \cdot G_y \cdot y \cdot \Delta t$

Where:

ϕ is the phase shift.

G_y is the gradient strength along the y-axis.

y is the position along the y-axis.

Δt is the duration of the gradient application.

5. Fourier transform

5.1. Image reconstruction

Process: The acquired signals, which are complex waveforms resulting from the superposition of signals from different locations, are processed using the Fourier transform. This mathematical technique converts the time-domain signal into the frequency domain, allowing for the spatial distribution of the nuclei to be reconstructed.

Fourier Transform Equation: The Fourier transform $F(k)$ of a signal $f(t)$ can be expressed as:

$$F(k) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi kt} dt$$

Where:

$F(k)$ is the Fourier-transformed function in the frequency domain.

$f(t)$ is the time-domain signal.

k represents the spatial frequency.

Final image: The result of this transformation provides the spatial distribution of the nuclei, which is then processed to create the final image displayed to the radiologist.

The intricate processes of spatial encoding, slice selection, frequency and phase encoding, and Fourier transform are fundamental to the functionality of MRI technology. By manipulating these parameters, MRI can generate high-resolution images of internal structures, providing valuable diagnostic information in clinical practice. Understanding these concepts enhances the ability to optimize imaging protocols and improve diagnostic accuracy.

5. MRI contrast

Magnetic Resonance Imaging (MRI) is a sophisticated imaging technique that utilizes the principles of nuclear magnetic resonance to generate detailed images of internal body structures. The contrast in MRI images is primarily influenced by the relaxation times of tissues, which are characterized by T1 (longitudinal) and T2 (transverse) relaxation. Below is an extended explanation of the various types of MRI contrast, including relevant equations.

5.1. T1-Weighted Images

Short TR and TE: T1-weighted images are produced using short Repetition Times (TR) and Echo Times (TE). This emphasizes differences in T1 relaxation times among tissues.

T1 Relaxation: The longitudinal relaxation time, T1, describes how quickly the net magnetization vector returns to its equilibrium state after being disturbed by an RF pulse. The equation governing T1 relaxation is given above.

Contrast: In T1-weighted images, fat appears bright due to its shorter T1 relaxation time compared to water, which appears darker.

5.2. T2-Weighted Images

Long TR and TE: T2-weighted images are obtained using long TR and TE, which highlights differences in T2 relaxation times.

T2 Relaxation: The transverse relaxation time, T2, describes how quickly the transverse magnetization decays after the RF pulse. The equation for T2 relaxation is given above:

Contrast: In T2-weighted images, fluid-filled structures like cerebrospinal fluid appear bright due to their longer T2 relaxation times, while fat appears darker.

5.3. Proton Density-Weighted Images

Intermediate TR and TE: Proton density-weighted images utilize intermediate TR and TE values. These images focus on the density of hydrogen protons in tissues.

Proton Density (PD): The contrast is largely determined by the number of hydrogen protons in a given volume, which can be represented as:

$$PD = \frac{Number\ of\ protons}{Volume}$$

Contrast: High proton density results in brighter images, providing excellent spatial resolution and contrast between tissues with varying proton densities.

Contrast Agents

Gadolinium-based Agents: These agents are injected into the body to enhance image contrast, particularly in T1-weighted images.

Mechanism: Gadolinium shortens the T1 relaxation time in tissues where it accumulates. The effect can be described as:

$$T1' = T1 \cdot (1 - R_1)$$

Where:

$T1'$ is the modified T1 relaxation time in the presence of gadolinium,

R_1 is the relaxivity of the contrast agent, which is tissue-specific.

Impact on Imaging: The result is increased signal intensity in tissues with gadolinium, enhancing their visibility on T1-weighted images.

MRI is a powerful imaging modality that provides exceptional contrast between different types of tissues without utilizing ionizing radiation. By manipulating magnetic fields and radiofrequency pulses, MRI can differentiate between various tissues based on their T1 and T2 relaxation times, as well as hydrogen proton density. The use of contrast agents like gadolinium further enhances the diagnostic capabilities of MRI, allowing for improved visualization of specific structures and abnormalities.

Note: A visual representation of MRI contrast mechanisms can be added as Figure 7 to illustrate the differences in T1 and T2 relaxation times and the effects of contrast agents.

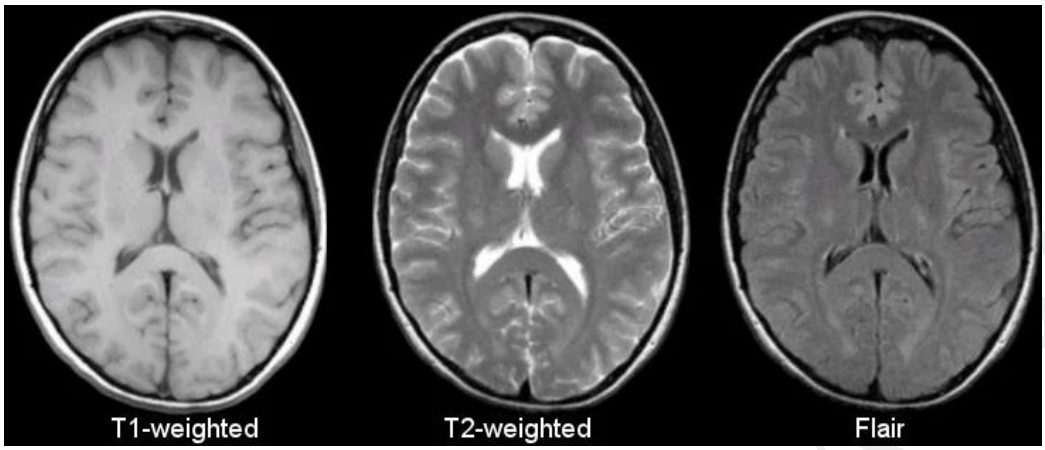


Figure 7: MRI contrast

6. Mathematical concepts of MRI

The mathematical principles of MRI encompass various operations that convert raw data obtained during an MRI scan into detailed images. This involves the physics of magnetic resonance, signal encoding and acquisition, and methods for image reconstruction.

6.1. k-Space

Definition: k-space is a mathematical space where the MRI signal data is stored during acquisition. Each point in k-space corresponds to a specific spatial frequency in the resulting image.

Fourier Transform Relationship: The spatial MRI image $I(x, y)$ is derived from the k-space data $S(k_x, k_y)$ through the inverse Fourier transform:

$$I(x,y) = \int \int S(kx,ky) e^{i2\pi(kxx+kyy)} dkxdky$$

Where:

$S(k_x,k_y)$ is the signal in k-space,

(x,y) are the spatial coordinates in the image,

(k_x,k_y) are the spatial frequencies in k-space.

k-Space Sampling: The sampling of k-space can be described using the Nyquist theorem, which states that the sampling frequency must be at least twice the highest frequency component present in the signal

$$f_s \geq 2f_{max}$$

Where f_s is the sampling frequency, and f_{max} is the maximum frequency present in the signal.

Spatial Frequency Representation: The relationship between spatial coordinates and spatial frequencies can be further explored through the Fourier transform pairs:

$$S(kx, ky) = \int \int I(x, y) e^{-i2\pi(kxx+kyy)} dxdy$$

The mathematical concepts of MRI are critical for understanding how images are generated from raw data. By employing the Bloch equations, Fourier transform relationships, and the principles of k-space, MRI is able to produce high-resolution images essential for medical diagnostics. Understanding these equations and their implications allows for the optimization and advancement of MRI technology.

This concept includes several operations that generate detailed images from the data attained during an MRI scan. Indeed, the physics of magnetic resonance, the principles of signal encoding and acquisition, and the methods of image reconstruction are the keys of this concept.

6.2. Signal acquisition

6.2.1. Magnetization and precession:

Net magnetization vector $\overrightarrow{M}$: The sum of the magnetic moments of the nuclei in the sample. In the presence of an external magnetic field B_0, the net magnetization vector processes around the direction of B_0.

Bloch equations: Describe the dynamics of the net magnetization vector under the influence of magnetic fields and relaxation processes:

$$dM/dt = \gamma(M \times B) - (\frac{M_x}{T_2})\vec{x} - (\frac{M_y}{T_2})\vec{y} - (\frac{M_z - M_0}{T_1})\vec{z}$$

Where:

- B is the total magnetic field; T_1 and T_2 are the longitudinal and transverse relaxation time; γ is the gyromagnetic ratio.

Components of Magnetization: The components of the magnetization vector can be expressed as:

$$M_x = M_0 sin(\theta) cos(\phi)$$

$$M_z = M_0 cos(\theta)$$

Where θ is the angle of the magnetization vector with respect to the z-axis, and ϕ is the azimuthal angle.

Larmor Frequency: The frequency at which the net magnetization precesses around the magnetic field is given by:

$$\omega_0 = \gamma B_0$$

This frequency is crucial for determining the timing of RF pulses and signal acquisition.

6.2.2. k-Space:

Definition: k-space is a mathematical space where the MRI signal data is stored during acquisition. Each point in k-space corresponds to a specific spatial frequency in the image.

Fourier transform relationship: The spatial MRI image $I(x, y)$ is the inverse Fourier transform of the k-space data $S(k_x, k_y)$:

- $I(x, y) = \int \int S(k_x, k_y) e^{i2\pi(k_x x + k_y y)} dk_x dk_y$

Where:

$S(k_x, k_y)$ is the signal in k-space.

(x, y) are the spatial coordinates in the image.

(k_x, k_y) are the spatial frequencies in k-space.

k-Space Sampling: The sampling of k-space can be described using the Nyquist theorem, which states that the sampling frequency must be at least twice the highest frequency component present in the signal:

$$fs \geq 2fmax$$

Where f_s is the sampling frequency, and f_{max} is the maximum frequency present in the signal.

Spatial Frequency Representation: The relationship between spatial coordinates and spatial frequencies can be further explored through the Fourier transform pairs:

$$S(kx, ky) = \int \int I(x, y) e^{-i2\pi(kxx + kyy)} dxdy$$

The mathematical concepts of MRI are critical for understanding how images are generated from raw data. By employing the Bloch equations, Fourier transform relationships, and the principles of k-space, MRI is able to produce high-resolution images essential for medical diagnostics. Understanding these equations and their implications allows for the optimization and advancement of MRI technology.

7. Spatial encoding

7.1. Gradient fields:

Gradient coils: Apply linearly varying magnetic fields along the x, y, and z axes. These gradients cause the Larmor frequency to vary with position, enabling spatial encoding.

Gradient strength: The gradient field strength $\vec{G}$ determines the rate of change of the magnetic field with position:

$$B(x) = B_0 + G_x x + G_y y + G_z z$$

7.2. Frequency encoding:

Gradient application: During signal readout, a gradient is applied along one axis (e.g., the x-axis). This causes the precession frequency of the spins to vary linearly with position:

$$\omega(x) = \gamma(B_0 + G_x x)$$

Signal acquisition: The signal $S(t)$ is recorded as a function of time t:

$$S(t) = \int_{-\infty}^{\infty} \rho(x) e^{i(\omega_0 + \gamma G_x x)t}\, dx$$

- Where $\rho(x)$ is the spin density distribution.

7.3. Phase encoding:

Gradient Application: Prior to signal readout, a gradient is briefly applied along a perpendicular axis (e.g., the y-axis). This gradient induces a position-dependent phase shift in the spins, which can be expressed as:

$$\phi(y) = \gamma G_y y \Delta t$$

Where:

Δt is the duration of the gradient pulse.

Signal with Phase Encoding: The resulting signal, which incorporates both frequency and phase encoding, can be described by the equation:

$$S(t, \Delta t) = \int \int \rho(x, y) e^{i(\omega_0 + \gamma G_x x)t} e^{i G_y y \Delta t}\, dx\, dy$$

Note: A visual representation of k-space and image space is typically represented in figure 7. this section to illustrate the relationship between the spatial encoding process, k-space data acquisition, and the resulting image reconstruction.

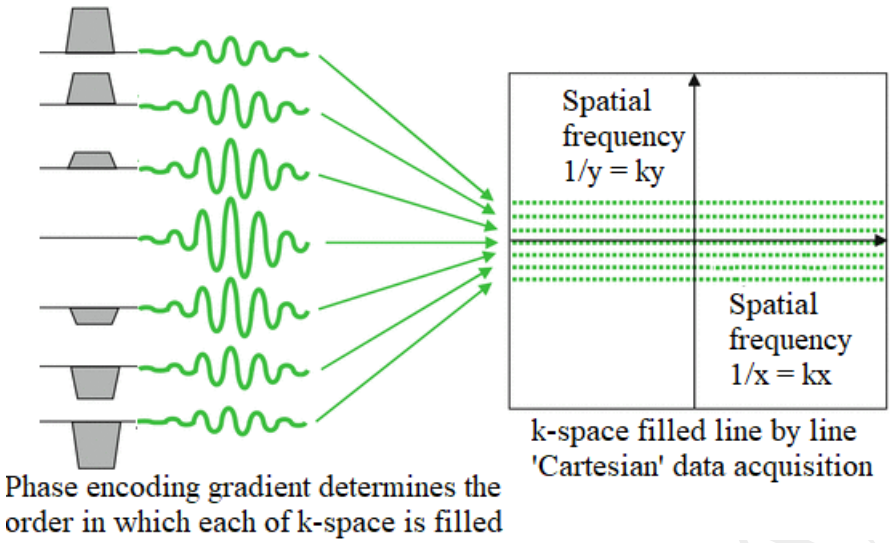


Figure 7: K-space and image space

Spatial encoding in MRI is achieved through the use of gradient fields that modify the magnetic field strength and, consequently, the Larmor frequency of the spins. By applying these gradients in specific directions (frequency encoding along one axis and phase encoding along another), MRI can accurately reconstruct images of the internal structures of the body. The mathematical equations governing these processes provide the framework for understanding how spatial information is encoded in the acquired signals, ultimately leading to high-resolution images essential for medical diagnostics.

8. Image reconstruction

8.1. Inverse Fourier transform:

- **Reconstructing the image**: The MRI image is reconstructed by applying the inverse Fourier transform to the k-space data:
- $I(x,y) = \int\int S(k_x,k_y)e^{i2\pi(k_x x+k_y y)}dk_x dk_y$

8.2. Discrete Fourier transform (DFT):

- **Sampling**: In practice, k-space data is sampled at discrete intervals, and the discrete Fourier transform (DFT) is used for reconstruction.
- **DFT formula**: For a 2D image, the DFT is given by:

$$I(m,n)\sum_{p=0}^{N-1}\sum_{q=0}^{M-1}S(p,q)e^{i2\pi(\frac{mp}{N}-\frac{nq}{M})}$$

- Where: $S(p,q)$ are the discrete k-space samples; (m,n) are the pixel indices in the image; N and M are the dimensions of the image.

8.3. Fast Fourier transform (FFT):

Efficiency: The Fast Fourier Transform (FFT) algorithm is used to efficiently compute the DFT, reducing the computational complexity from $O(N^2)$ to $O(Nlog\,N)$.

The process of image reconstruction in MRI relies heavily on the principles of Fourier analysis. By applying the inverse Fourier transform to k-space data, the MRI system can generate images that accurately represent the internal structures of the body. The use of the discrete Fourier transform allows for the handling of sampled data, while the Fast Fourier Transform enhances computational efficiency, making real-time imaging and analysis possible. Understanding these mathematical foundations is essential for optimizing MRI techniques and improving image quality. We can also note additional notes reconstruction technics such as:

Windowing and Filtering: Techniques such as windowing and filtering may be applied during reconstruction to reduce artifacts and enhance image quality. For example, applying a Hamming window can minimize spectral leakage in the DFT.

Regularization Techniques: In some cases, regularization methods may be employed to improve the stability of the reconstruction process, especially in cases of undersampled k-space data.

Parallel Imaging: Advanced reconstruction techniques like parallel imaging exploit multiple receiver coils to accelerate data acquisition and improve image quality by reducing scan times. That we give more details on following paragraph.

By integrating these concepts and techniques, MRI continues to evolve, providing clearer and more detailed images critical for accurate diagnosis and treatment planning.

9. Advanced concepts

Advanced imaging techniques have significantly enhanced the capabilities of MRI, allowing for faster acquisition times, improved image quality, and the ability to capture dynamic processes. This section discusses three key advanced concepts: parallel imaging, compressed sensing, and echo planar imaging (EPI).

9.1. Parallel imaging:

Multiple Coils: Parallel imaging utilizes multiple receiver coils to simultaneously acquire data from different spatial locations. This approach significantly accelerates image acquisition by collecting k-space data from several points at once.

Mathematical Representation: If $S(k_x,k_y)$ represents the k-space data acquired by a single coil, the total k-space data total $S_{total}(k_x,k_y)$ from N coils can be expressed as:

$$Stotal(kx, ky) = \sum_{i=1}^{N} Si(kx, ky)$$

Where $S_i(k_x,k_y)$ is the k-space data from the i-th coil.

Sensitivity encoding (SENSE): SENSE is a widely used parallel imaging technique that exploits the unique sensitivity profiles of each coil to reconstruct images from undersampled k-space data.

Reconstruction equation: The SENSE reconstruction process can be described mathematically as follows:

$$I(x,y) = \sum_{i=1}^{N} S_i(k_x, k_y) \cdot Sensitivity_i(x,y)$$

Where:

$\text{Sensitivity}_i(x,y)$ is the spatial sensitivity profile of the i-th coil,

$I(x,y)$ is the reconstructed image.

Undersampling Factor: The acceleration factor R indicates how many times faster the acquisition is compared to traditional methods, allowing for fewer samples in k-space:

$$R = \frac{Number\ of\ samples\ in\ traditional\ imaging}{Number\ of\ samples\ in\ parallel\ imagin}$$

Multiple coils: Uses multiple receiver coils to acquire data simultaneously, accelerating image acquisition.

Sensitivity encoding (SENSE): One common parallel imaging technique that uses the sensitivity profiles of the coils to reconstruct the image from undersampled k-space data.

9.2. Compressed sensing:

Sparse Sampling: Compressed sensing is a technique that allows for the acquisition of fewer k-space samples than traditional methods by exploiting the sparsity of images in a certain transform domain (e.g., wavelet or Fourier domain).

Mathematical Framework: The core idea is that a signal can be reconstructed from a small number of measurements if it is sparse in some domain. The reconstruction problem can be formulated as:

$$min\ ||\ I\ ||_1\ subject\ to\ \ ||\ \Phi I - y\ ||_2 \leq\ \epsilon$$

Where:

$||I||_1$ is the L_1 norm promoting sparsity,

Φ is the transformation matrix,

y is the undersampled k-space data,

ϵ is a tolerance level.

Reconstruction Algorithms: Various algorithms, such as Iterative Shrinkage-Thresholding Algorithm (ISTA) and Total Variation (TV) minimization, are employed to solve this optimization problem efficiently.

Sparse sampling: Acquires fewer k-space samples than traditional methods and uses mathematical algorithms to reconstruct the image, exploiting the sparsity of the image in some transform domain.

9.3. Echo planar imaging (EPI):

Rapid Acquisition: EPI is a fast imaging technique that acquires an entire 2D image from a single excitation. It achieves this by rapidly switching gradient fields, allowing for the collection of multiple lines of k-space data in a single echo.

Mathematical Description: The k-space trajectory for EPI can be described by a series of gradient pulses that create a spiral or zigzag pattern in k-space. The relationship between the readout time t and the spatial frequency k can be expressed as:

$$k_x(t) = \gamma G_x t$$

Where:

γ is the gyromagnetic ratio,

G_x is the gradient strength along the x-axis.

Applications: EPI is particularly useful in functional MRI (fMRI) and diffusion MRI, where rapid imaging is essential for capturing dynamic physiological processes.

Rapid acquisition: Acquires an entire 2D image from a single excitation by rapidly switching gradient fields, often used in functional MRI and diffusion MRI. See figure 8.

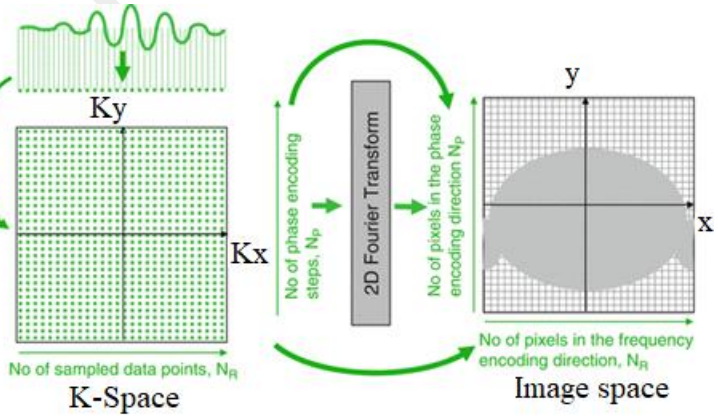


Figure 8: Image reconstruction

Advanced imaging techniques such as parallel imaging, compressed sensing, and echo planar imaging enhance the capabilities of MRI by enabling faster acquisition times and improved image

quality. These methods leverage mathematical principles and algorithms to reconstruct high-resolution images from limited data, making them invaluable tools in modern medical imaging.

Chapter 3. Ultrasound imaging

1. Introduction

Ultrasound imaging, also known as sonography, serves as a non-invasive medical imaging technique that uses high-frequency sound waves to create detailed images of the body's internal structures. This technology relies on an ultrasound machine that generates sound waves. A handheld device called a transducer sends these sound waves into the body. As the sound waves travel through different tissues and organs, they bounce back, producing echoes that vary in intensity depending on the type of tissue encountered. The transducer captures these echoes and sends them back to the ultrasound machine, which processes the information and converts it into visual images displayed on a monitor. This process allows healthcare professionals to visualize and assess the condition of various organs and tissues in real time.

Ultrasound imaging encompasses three primary types: 2D, 3D, and 4D ultrasound. The 2D ultrasound produces flat, two-dimensional images and commonly helps practitioners monitor fetal development in obstetrics. In contrast, the 3D ultrasound provides three-dimensional images, offering enhanced detail and depth, which benefits for more comprehensive prenatal assessments. Finally, the 4D ultrasound builds on the 3D technology by adding the element of time, enabling healthcare providers to observe real-time movement and activity within the womb. This dynamic imaging capability significantly enhances the understanding of fetal behavior and development.

Healthcare professionals use ultrasound imaging across various medical fields due to its versatility and effectiveness. In obstetrics and gynecology, practitioners monitor fetal development, assess maternal health, and evaluate reproductive organs. In cardiology, ultrasound plays a crucial role in assessing heart function and diagnosing conditions affecting the heart's structure. Additionally, abdominal imaging utilizes ultrasound to evaluate vital organs such as the liver, kidneys, and pancreas for abnormalities. Musculoskeletal imaging also benefits from ultrasound, allowing practitioners to examine muscles, tendons, and joints for injuries or underlying conditions.

One of the key advantages of ultrasound imaging lies in its non-invasive nature. This technique eliminates the need for incisions or injections, making it a safer option for patients. Furthermore, ultrasound provides real-time measurements, enabling healthcare professionals to observe moving structures, such as a beating heart or a developing fetus. This imaging modality does not use ionizing radiation, which enhances its safety profile for both patients and technicians. Additionally, practitioners can perform ultrasound at the bedside or in various clinical settings, making it a convenient option for immediate assessments.

Despite its many benefits, ultrasound imaging presents some limitations. Image quality can suffer due to factors such as obesity or the presence of gas in the intestines, which may obstruct sound wave transmission. Moreover, the technique may not penetrate deeply enough to visualize certain structures clearly, limiting its effectiveness in specific areas. Understanding these limitations helps healthcare professionals make informed decisions regarding the appropriate use of ultrasound in diagnostic imaging.

2. Sonography concept

The mathematical concepts behind ultrasound imaging are crucial for biomedical engineers, providing a framework to understand the complex interactions between sound waves and biological tissues. Utilizing high-frequency sound waves, ultrasound technology allows for non-invasive visualization of internal structures, making it an essential tool across various medical specialties.

Central to ultrasound imaging is the interaction of sound waves with different tissue types, each reflecting and absorbing waves uniquely based on their density and acoustic properties. For biomedical engineers, grasping concepts such as wave speed, frequency, and impedance is vital. For example, the speed of sound in human tissue averages around 1540 m/s, a factor that influences how sound waves travel and return to the transducer, affecting the accuracy of depth and structural assessments.

Signal processing also plays a pivotal role. The system must interpret echoes to create coherent images, utilizing complex algorithms to calculate distances and construct two-dimensional or three-dimensional representations. Advanced techniques like Doppler ultrasound enhance this process by measuring frequency shifts from moving objects, offering insights into blood flow dynamics that are critical for cardiovascular diagnostics.

Additionally, harmonic imaging leverages the nonlinear properties of sound wave propagation to enhance image quality and reduce artifacts. Understanding these mathematical principles is essential for biomedical engineers focused on optimizing ultrasound technologies and improving diagnostic capabilities.

In summary, the mathematical foundations of ultrasound imaging are fundamental for biomedical engineers, enabling them to innovate and refine ultrasound applications, ultimately leading to better patient outcomes and advancements in medical diagnostics

2.1. Sound wave propagation and reflection

2.1.1. Speed of sound

The speed of sound in a given medium plays a crucial role in calculating the distance between the transducer and the tissues. In human tissue, the speed of sound measures approximately 1540 m/s. However, this speed varies based on the type of tissue and its density. For instance, sound travels faster in denser tissues, such as bone, compared to softer tissues like fat or fluid. You can summarize the speed of sound in various tissues as follows:

Fat	Muscle	Blood	Bone
~1450 m/s	~1580 m/s	~1570 m/s	~4080 m/s

These variations are important for accurate imaging, as they affect the timing of the echoes received by the transducer.

2.1.2. Time-of-flight principle

The time it takes for an ultrasound pulse to travel to a tissue and back is used to calculate the distance to that tissue. This principle is known as the time-of-flight principle. The distance d is computed using the formula:

$$d = 2vt$$

Where:

d is the distance to the tissue,

v is the speed of sound in the tissue,

t is the time it takes for the echo to return.

This formula accounts for the round trip of the sound wave, hence the factor of 2. See figure 1.

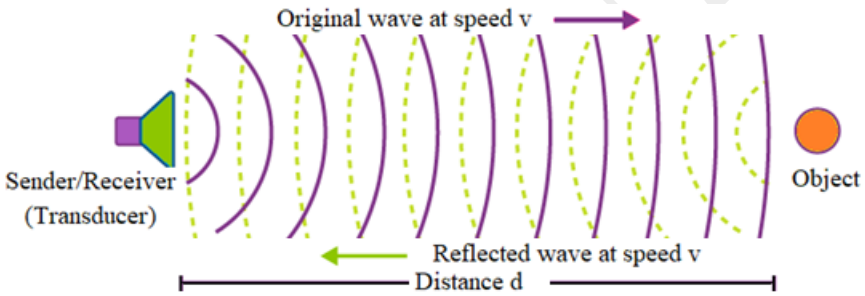


Figure 1: Sound propagation

In this figure, sound waves are illustrated as they propagate through various tissues, reflecting off boundaries and returning to the transducer. The dynamics of sound wave propagation and reflection are essential for accurate imaging and diagnosis in ultrasound technology.

2.1.3. Reflection and Impedance

When sound waves encounter a boundary between two different tissues, some of the energy is reflected back while the rest is transmitted into the next medium. The reflection coefficient R quantifies the proportion of the sound wave that is reflected at the interface of two tissues and can be calculated using the acoustic impedances Z_1 and Z_2 of the two tissues:

$$R = \left(\frac{Z_2 - Z_1}{Z_2 + Z_1}\right)^2$$

Where:

Z_1 is the acoustic impedance of the first medium,

Z_2 is the acoustic impedance of the second medium.

The acoustic impedance Z is defined as:

$$Z = \rho v$$

Where:

ρ is the density of the medium,

v is the speed of sound in that medium.

2.1.4. Total Time Calculation

To further elaborate on the time-of-flight principle, if we consider multiple tissue boundaries, the total time T for the sound to travel through several layers can be expressed as:

$$T = \sum_{i=1}^{n} \frac{2d_i}{v_i}$$

Where:

T is the total time,

d_i is the distance to each tissue layer,

v_i is the speed of sound in each corresponding tissue layer,

n is the number of tissue boundaries encountered.

By understanding these principles and equations, biomedical engineers and healthcare professionals can enhance the effectiveness of ultrasound imaging, leading to improved diagnostic accuracy and patient care.

2.2. Image resolution and depth

2.2.1. Axial resolution

Axial resolution refers to the ability to distinguish between two points along the axis of the ultrasound beam. It is determined by the pulse duration Δt and the frequency f of the transducer. The axial resolution AR can be estimated as:

$$AR = \frac{c}{2f}$$

Where:

c is the speed of sound in the tissue,

f is the frequency of the ultrasound.

Higher frequencies provide better axial resolution but have less penetration depth. See figure 2.

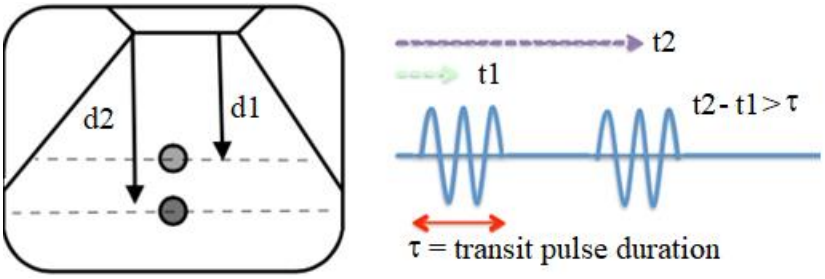


Figure 2: Axial resolution

2.2.2. Lateral resolution

Lateral resolution refers to the ability to distinguish between two points perpendicular to the axis of the ultrasound beam. It is influenced by the width of the ultrasound beam and the focusing of the beam. The lateral resolution LR can be approximated as:

$$LR = \frac{D}{2}$$

Where:

D is the diameter of the ultrasound beam at the depth of interest. See figure 3.

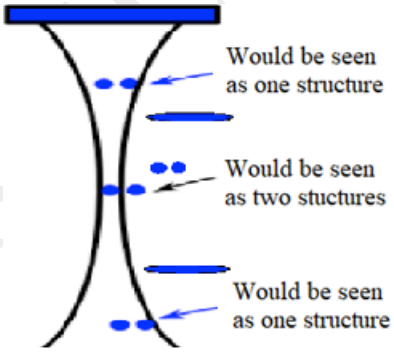


Figure 3: Lateral resolution

2.2.3. Beamforming

Beamforming is the process of directing the ultrasound beam to focus on specific regions. Mathematically, it involves adjusting the timing of the signals received from different elements of the transducer array to construct a focused image. The time delay τ for each element is given by:

$$\tau = \frac{d\sin\theta}{v}$$

Where:

d is the distance between the transducer elements,

θ is the desired angle of focus,

v is the speed of sound in the material.

By summing the signals from different transducer elements with these appropriate time delays, the system enhances signals from a specific direction, resulting in improved image quality. See figure 4 and figure 5.

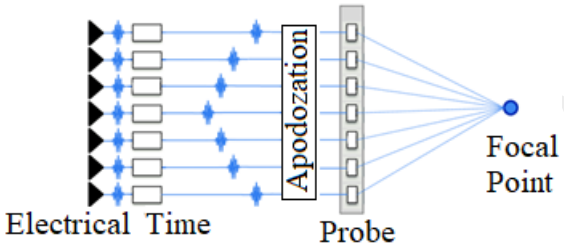


Figure 4: Beamforming signal

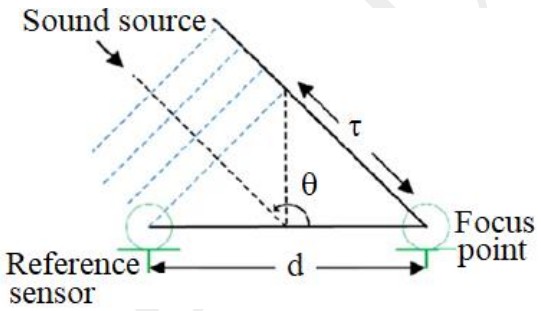


Figure 5: Beamforming signal

3. Doppler-effect

Doppler ultrasound utilizes the Doppler effect to measure the change in frequency of the sound waves reflected from moving objects, such as blood cells. The frequency shift Δf can be calculated using:

$$\Delta f = \frac{2 f_0 v \cos\theta}{V_s}$$

Where:

f_0 is the original frequency of the ultrasound,

v is the velocity of the moving object (e.g., blood flow),

θ is the angle between the ultrasound beam and the direction of the moving object,

V_s is the speed of sound in the medium.

This equation forms the basis for assessing blood flow and detecting abnormalities in cardiovascular diagnostics. See figure 6.

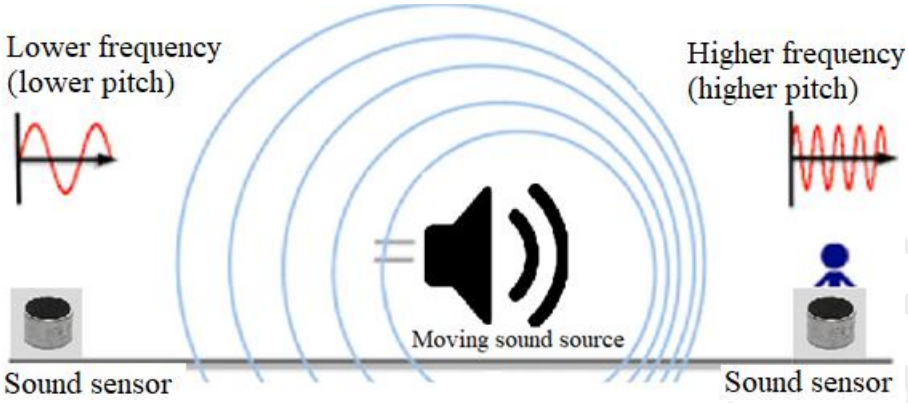


Figure 6: Doppler Effect

Example: The change in pitch of a passing ambulance siren is a common example of the Doppler Effect. As the ambulance approaches, the siren's pitch increases; as it moves away, the pitch decreases.

4. Concept and application in medical imaging

Medical imaging is a crucial field that encompasses various techniques and technologies used to visualize the interior of the body for clinical analysis and medical intervention. This discipline plays a vital role in diagnosing diseases, monitoring treatment progress, and guiding surgical procedures. The primary modalities include X-ray, computed tomography (CT), magnetic resonance imaging (MRI), ultrasound, and nuclear medicine. Each of these techniques offers unique advantages and applications, contributing to a comprehensive understanding of human anatomy and pathology.

For biomedical engineers, the intersection of engineering principles and medical imaging technology presents exciting opportunities. They are instrumental in developing advanced imaging systems, improving image quality, and enhancing diagnostic capabilities. Biomedical engineers work on creating innovative algorithms for image reconstruction, developing contrast agents, and integrating artificial intelligence to automate and refine image analysis. Their contributions are essential in advancing personalized medicine, improving patient outcomes, and reducing healthcare costs, making medical imaging a dynamic and impactful area of focus within biomedical engineering.

4.1. Basic principle

The Doppler Effect is a phenomenon observed when there is a relative motion between a wave source and an observer. It is particularly significant in medical imaging, where it is used to assess the velocity of moving structures, such as blood flow in vessels. The effect can be observed in various types of waves, including sound and electromagnetic waves.

4.1.1. Frequency Shift

When a wave source moves towards an observer, the observed frequency increases, resulting in a higher pitch for sound waves or a higher frequency for light waves. Conversely, when the source moves away from the observer, the observed frequency decreases, leading to a lower pitch or frequency.

- **Doppler shift:** The Doppler Effect is used to measure the velocity of moving structures, such as blood flow. It involves analyzing the change in frequency of the returned echoes relative to the transmitted frequency:

- The Doppler shift Δf, known as the **Frequency Shift**: When a wave source moves towards an observer, the frequency of the wave increases, leading to a higher pitch (in sound waves) or a higher frequency (in light waves). Conversely, if the source moves away from the observer, the frequency decreases, resulting in a lower pitch or lower frequency. Δf is calculated as:

$$\Delta f = \frac{2 f_0 v \cos(\theta)}{C}$$

- Where : f_0 is the emitted frequency of the ultrasound wave; v is the velocity of the moving object (e.g., blood); θ is the angle between the ultrasound beam and the direction of motion; c is the speed of sound in the medium.

4.1.2. Observed frequency

The observed frequency f' can be calculated using the following formula

$$f' = f(v + v_o)/(v - v_s)$$

Where:

f = original frequency of the wave;

v speed of the wave in the medium;

v_o speed of the observer relative to the medium;

v_s = speed of the source relative to the medium

This formula reflects the motion of both the source and the observer. The sign depends on whether they are moving towards or away from each other.

4.1.3. Interpretation

Observer moving towards the source: If the observer is moving towards the source, v_o is positive, leading to an increase in the observed frequency.

Observer moving away from the source: If the observer is moving away, v_o is negative, resulting in a decrease in the observed frequency.

Source moving towards the observer: If the source moves towards the observer, v_s is negative, which also increases the observed frequency.

Source moving away from the observer: If the source moves away, v_s is positive, leading to a decrease in the observed frequency.

4.1.4. Example

Consider an ultrasound wave with the following parameters:

Emitted frequency, f_0=5MHz

Speed of sound in tissue, c=1540m/s

Velocity of blood flow, v=0.5m/s

Angle, θ=0° (directly towards)

Using the Doppler shift formula:

$$\Delta f = \frac{2 \times 5 \times 10^6 Hz \times \frac{0.5m}{s} \times \cos(0)}{\frac{1540m}{s}}$$

$$= 15405 \times 10^6 \times 0.5$$

$$\approx 1620.78Hz$$

This frequency shift can then be used to determine the velocity of the blood flow, demonstrating the practical application of the Doppler Effect in medical imaging.

The Doppler Effect is a fundamental principle in medical imaging that enables the measurement of velocities of moving structures, such as blood flow. Understanding the mathematical representations and their implications is crucial for biomedical engineers and medical professionals in applying these concepts effectively in clinical settings.

4.2. Application in medical imaging

Doppler ultrasound is a non-invasive imaging technique that leverages the Doppler effect to measure the velocity and direction of blood flow within the body. This technology is particularly valuable in assessing cardiovascular health, aiding in the detection of conditions such as blood clots, blocked arteries, and heart valve defects.

4.2.1. Color Doppler

Color Doppler ultrasound enhances the visualization of blood flow by employing color coding. In this technique:

- **Flow towards the transducer** is typically represented in **red.**
- **Flow away from the transducer** is indicated in **blue.**

This color-coding system allows clinicians to quickly assess both the direction and speed of blood flow, providing critical information for diagnosing various cardiovascular conditions. See figure 7.

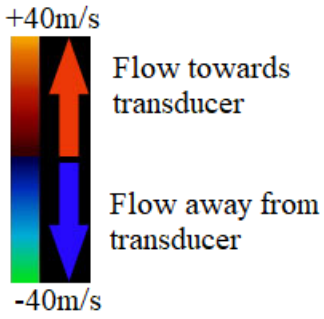


Figure 7: Color Doppler

4.2.2. Continuous Wave Doppler

Principle

Continuous wave Doppler utilizes two crystals within the transducer:

One crystal continuously **emits** ultrasound waves.

The other continuously **receives** the reflected waves.

This method is particularly effective for measuring high-velocity blood flow, such as in cases of stenosis (narrowing of blood vessels). We did see above the mathematical frequency shift.

Limitation

While continuous wave Doppler provides accurate measurements of blood flow velocity, it cannot localize the source of the signal, meaning it does not provide information on the exact location of the flow being measured.

4.2.3. Pulsed Wave Doppler

Principle

Pulsed wave Doppler sends out short bursts (pulses) of ultrasound waves and measures the frequency shift of the returning echoes. This technique allows for specific sampling at different depths, making it useful for measuring blood flow in targeted areas. You can rvise the paragraph before about the observed frequency that allows for the calculation of the frequency shift based on the relative motion of the source and observer.

Limitation

Pulsed wave Doppler is limited in its ability to measure very high velocities due to a phenomenon known as **aliasing**. Aliasing occurs when the frequency shift exceeds half the pulse repetition frequency (PRF), leading to misinterpretation of the flow direction or speed. See figure 8.

Doppler ultrasound, through its various applications like Color Doppler, Continuous Wave Doppler, and Pulsed Wave Doppler, plays a crucial role in modern medical imaging. By utilizing the Doppler effect, these techniques provide valuable insights into blood flow dynamics, aiding in the diagnosis and management of cardiovascular diseases. Understanding the underlying principles and mathematical representations enhances the ability of healthcare professionals to interpret results accurately and make informed clinical decisions.

Doppler ultrasound is a non-invasive imaging technique that uses the Doppler effect to measure the velocity and direction of blood flow in the body. It is particularly useful in assessing cardiovascular health, including the detection of blood clots, blocked arteries, and heart valve defects.

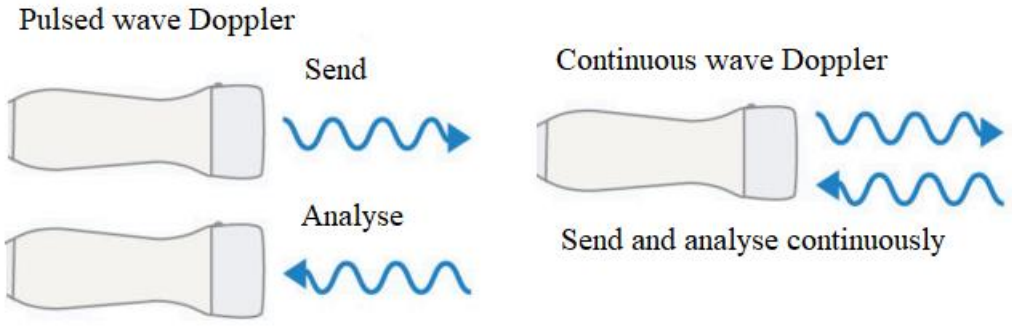


Figure 8: Pulse and continuous wave Doppler principal

4.3. Applications of Doppler-effect in medicine

4.3.1. Cardiovascular Assessment

Blood flow measurement: Doppler ultrasound is instrumental in measuring the velocity of blood flow in arteries and veins. For example, in diagnosing **deep vein thrombosis (DVT)**, a clinician may use Doppler ultrasound to assess blood flow in the femoral vein. If the ultrasound detects a significantly reduced velocity or a complete absence of flow, it may indicate the presence of a thrombus (blood clot). Similarly, for **peripheral artery disease (PAD)**, Doppler ultrasound can measure the blood flow in the popliteal artery to evaluate the severity of the condition.

Heart function: In echocardiography, the Doppler effect is utilized to evaluate heart valve function and measure cardiac output. For instance, in a patient with suspected **aortic stenosis**, Doppler ultrasound can be employed to measure the velocity of blood flow across the aortic valve. Using the continuity equation, clinicians can calculate the aortic valve area:

$$A_1V_1 = A_2V_2$$

Where A_1 and V_1 are the cross-sectional area and velocity of blood flow in the left ventricle, and A_2 and V_2 are the area and velocity across the aortic valve. This calculation helps in assessing the severity of the stenosis.

4.3.2. Maternal and fetal health monitoring

Fetal monitoring: Doppler ultrasound is routinely employed to monitor fetal heart rate and blood flow in the umbilical artery. For example, during a routine prenatal check-up, a clinician may use Doppler ultrasound to assess the umbilical artery's blood flow. By measuring the **systolic/diastolic ratio (S/D ratio)**, clinicians can evaluate fetal well-being. An elevated S/D ratio may indicate placental insufficiency, prompting further investigation or intervention.

4.3.3. Vascular surgery

Pre- and post-operative evaluation: Doppler ultrasound is vital before and after vascular surgeries. For instance, after a **bypass graft surgery**, Doppler ultrasound can be used to assess blood flow in the graft. A normal flow pattern with adequate velocity indicates successful graft placement. Conversely, a significant decrease in flow velocity may suggest graft occlusion or stenosis, necessitating further surgical intervention.

4.4. Advantages and limitations

4.4.1. Advantages

Non-invasive: Doppler ultrasound provides valuable information without the need for invasive procedures, making it a patient-friendly option for assessing cardiovascular health.

Real-time Imaging: This technique allows for real-time assessment of blood flow and heart function, making it highly useful in emergency and dynamic situations, such as evaluating a patient with chest pain for possible myocardial infarction.

Wide Range of Applications: The versatility of the Doppler effect extends across various areas of medicine, including cardiology, obstetrics, and vascular medicine, providing comprehensive diagnostic capabilities.

4.4.2. Limitations

Angle Dependence: The accuracy of Doppler measurements is highly dependent on the angle between the ultrasound beam and the direction of blood flow. For example, if the angle exceeds 60 degrees, the measured velocity can be significantly underestimated, leading to potential misdiagnosis.

Aliasing: In pulsed wave Doppler, aliasing can occur at high velocities, resulting in misinterpretation of the flow data. For example, in a patient with severe aortic regurgitation, the high diastolic velocities may lead to confusion in interpreting the flow pattern.

Limited by Penetration Depth: Ultrasound waves may not penetrate deeply enough in certain patients, such as those with high body mass index (BMI). This limitation can restrict the

effectiveness of Doppler imaging, particularly in obese patients where deeper structures may not be adequately visualized.

The applications of the Doppler effect in medicine, particularly in cardiovascular assessment, obstetrics, and vascular surgery, illustrate its critical role in modern diagnostic practices. While the advantages of Doppler ultrasound make it an invaluable tool, understanding its limitations is essential for accurate interpretation and effective patient management.

Chapter 4. Nuclear medicine

1. Introduction

Nuclear medicine involves the use of radioactive substances to diagnose and treat diseases. The primary mathematical concepts in nuclear medicine pertain to the physics of radioactive decay, the interaction of radiation with matter, the modeling of biological processes, and the reconstruction of images from the detected radiation.

Nuclear medicine imaging employs small amounts of radioactive material, a special camera, and a computer to create detailed images of the inside of the body. This technique provides information that is often unattainable through other imaging methods, making it invaluable in clinical settings. It helps diagnose various conditions, including many types of cancers, heart disease, gastrointestinal disorders, endocrine issues, and neurological disorders. Nuclear medicine procedures can pinpoint molecular activity within the body, allowing for the detection of diseases in their earliest stages when they are most amenable to treatment.

Biomedical engineers play a crucial role in advancing nuclear medicine technologies and applications. They actively engage in designing and optimizing imaging systems, ensuring that the equipment used in nuclear medicine is efficient, safe, and capable of producing high-quality images. By applying their expertise in imaging physics and engineering principles, biomedical engineers enhance the performance of gamma cameras and PET (positron emission tomography) scanners, enabling more accurate and precise diagnostics.

Moreover, biomedical engineers contribute to the development of novel radiopharmaceuticals, which are essential for targeted therapies and diagnostics. They leverage their understanding of biochemistry and molecular biology to design agents that can selectively bind to specific tissues or receptors, improving the specificity and effectiveness of treatments.

In addition, biomedical engineers work on the integration of advanced computational techniques and artificial intelligence in nuclear medicine. They develop algorithms for image reconstruction and analysis that enhance the interpretation of nuclear medicine scans, leading to better patient outcomes. By creating software solutions that facilitate the visualization and quantification of molecular activity, they empower clinicians to make more informed decisions.

Overall, the collaboration between nuclear medicine and biomedical engineering fosters innovation in diagnostic and therapeutic approaches, ultimately improving patient care and treatment efficacy.

2. Mathematical concepts

2.1. Radioactive decay

Radioactive decay is the process by which unstable atomic nuclei lose energy by emitting radiation. This phenomenon occurs in various forms, and understanding these types is crucial for

applications in nuclear medicine, radiation therapy, and radiological safety. There are three primary types of radioactive decay: see figure 1.

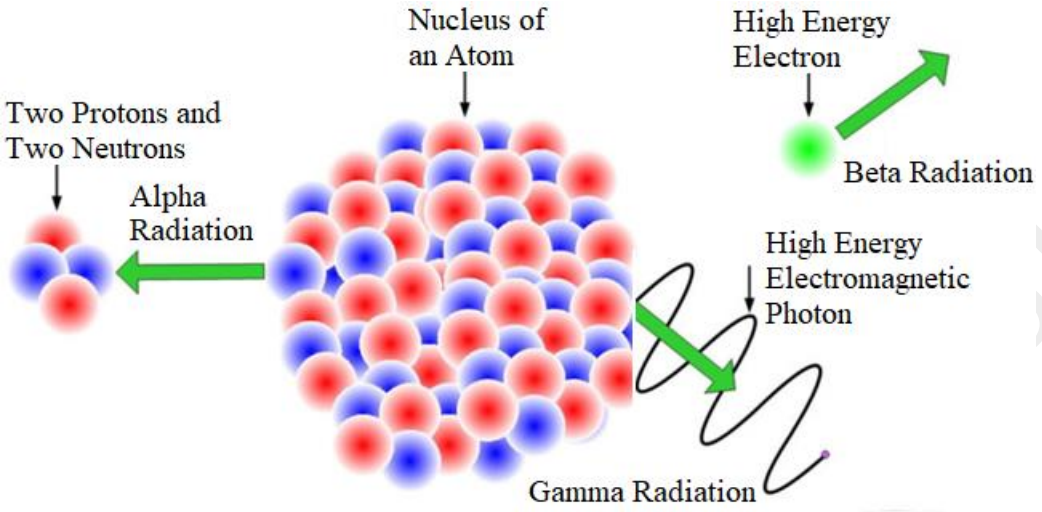


Figure 1: Three type of radioactive decay

2.1.1. Alpha Decay

In alpha decay, an unstable nucleus emits an alpha particle, which consists of two protons and two neutrons (essentially a helium nucleus). This process reduces the atomic number of the original atom by two and the mass number by four.

Example: A common example of alpha decay is the transformation of uranium-238 into thorium-234:

$$Uranium238 \rightarrow Thorium234 + \alpha\, particle$$

Applications: Alpha emitters are used in certain types of cancer treatments and in smoke detectors.

2.1.2. Beta Decay

Beta decay occurs when a neutron in the nucleus transforms into a proton (or vice versa), resulting in the emission of a beta particle. There are two types of beta decay:

Beta-minus decay: A neutron is converted into a proton, emitting an electron (beta particle) and an antineutrino.

Beta-plus decay (positron emission): A proton is converted into a neutron, emitting a positron (the antimatter counterpart of an electron) and a neutrino.

Example: A common example of beta-minus decay is the transformation of carbon-14 into nitrogen-14:

$$Carbon14 \rightarrow Nitrogen14 + \beta - \ particle + \bar{\nu}\ (\textbf{Antineutrino})$$

Applications: Beta emitters are widely used in medical imaging, such as in positron emission tomography (PET) scans, and in radiation therapy for cancer treatment.

2.1.3. Gamma Decay

Gamma decay involves the emission of gamma rays, which are high-energy photons, from an excited nucleus. This process typically follows alpha or beta decay, allowing the nucleus to transition from a higher energy state to a lower energy state without changing the number of protons or neutrons.

Example: After alpha decay, the resulting nucleus may be in an excited state and can emit a gamma photon:

$$Excited\ nucleus \rightarrow Stable\ nucleus + \gamma\ photon$$

Applications: Gamma rays are utilized in various medical applications, including cancer treatment (radiation therapy) and diagnostic imaging. They are also employed in sterilization processes and in the detection of radioactive materials.

This comprehensive overview of radioactive decay types highlights their significance in nuclear medicine and other fields, illustrating how they contribute to diagnostic and therapeutic applications.

2.2. Decay law**:**

2.2.1. Exponential decay

Radioactive decay follows an exponential decay law, which describes how the number of radioactive nuclei in a sample decreases over time. This phenomenon is a fundamental characteristic of radioactive materials and is crucial for understanding their behavior in various applications, including nuclear medicine, radiometric dating, and radiation safety.

The relationship governing the decay of radioactive nuclei can be expressed mathematically as:

$$N(t) = N_0 e^{-\lambda t}$$

Where:

N_0 is the initial number of nuclei. This value represents the total count of radioactive nuclei present in the sample at the start of the observation (time t=0). It is essential for calculating how many nuclei remain after a specific duration.

λ is the decay constant (specific to each radionuclide). This constant is a unique characteristic of each radionuclide, reflecting its stability and rate of decay. It is defined as the fraction of nuclei that decay per unit time. A higher decay constant indicates a faster decay rate.

The relationship between the decay constant and the half-life ($T_{1/2}$) of a radionuclide is given by:

$$T_{1/2} = \frac{\ln(2)}{\lambda} \approx \frac{0.693}{\lambda}$$

t is the time that represents the duration over which the decay is observed. As time increases, the number of remaining radioactive nuclei decreases exponentially.

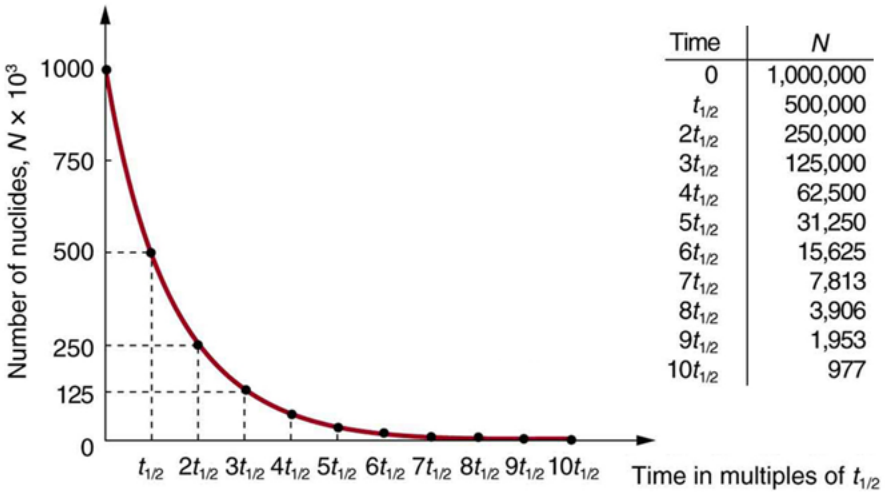
The decay process is continuous, meaning that at every moment, there is a probability that a nucleus will decay.

2.2.2. Characteristics

Rapid Initial Decline: The number of radioactive nuclei decreases quickly at first, but as time progresses, the rate of decay slows, leading to a gradual tapering off.

Half-Life Concept: The half-life is the time required for half of the radioactive nuclei in a sample to decay. This concept is essential for understanding the longevity of radioactive materials and is directly related to the decay constant.

For example, if a radionuclide has a half-life of 10 years, after 10 years, 50% of the original $N0$ will remain, after 20 years 25% will remain, and so forth. See figure 2.



Time	N
0	1,000,000
$t_{1/2}$	500,000
$2t_{1/2}$	250,000
$3t_{1/2}$	125,000
$4t_{1/2}$	62,500
$5t_{1/2}$	31,250
$6t_{1/2}$	15,625
$7t_{1/2}$	7,813
$8t_{1/2}$	3,906
$9t_{1/2}$	1,953
$10t_{1/2}$	977

Figure 2: Characteristic of exponential decay law

2.2.3. Applications of exponential decay

Radiometric Dating: Used in archaeology and geology to date ancient artifacts and geological formations based on the known decay rates of isotopes like Carbon-14 and Uranium-238.

Nuclear Medicine: Understanding the decay of radiopharmaceuticals helps in determining appropriate dosages and timing for diagnostic imaging and treatment.

Radiation Safety: Knowledge of decay rates is critical for managing exposure to radioactive materials and ensuring safety in environments where radiation is present.

The exponential decay law describes how the number of radioactive nuclei in a sample decreases over time, governed by the initial quantity, decay constant, and elapsed time. This law is fundamental to various scientific and medical applications, providing insight into the behavior of radioactive materials and their implications for health, safety, and research.

2.3. Activity

Definition: The activity (A) of a radioactive sample is the rate of decay, measured in Becquerel (Bq) or Curies (Ci). It is related to the number of nuclei (N) by the following equation:

$$A(t) = \lambda N(t) = A_0 e^{-\lambda t}$$

Where:

$A(t)$ is the activity at time t

λ is the radioactive decay constant, which represents the probability of a nucleus decaying per unit of time

$N(t)$ is the number of radioactive nuclei at time t

A_0 is the initial activity at time $t = 0$

The activity is a measure of the number of radioactive decays occurring in the sample per unit of time. The decay constant λ is a characteristic of the specific radioactive isotope and determines the rate of decay.

The exponential term $e^{-\lambda t}$ represents the fraction of the initial number of radioactive nuclei that remain at time t. As time passes, the activity decreases exponentially due to the radioactive decay. Note these key points about radioactive activity:

Units: The activity is measured in Becquerel (Bq), which represents one decay per second, or in Curies (Ci),

where $1\ Ci = 3.7 \times 10^{10} Bq.$

Half-life: The half-life ($t_1/_2$) is the time it takes for the activity to decrease to half of its initial value. It is related to the decay constant by the equation:

$t_1/_2 = ln(2) / \lambda.$

Specific Activity: The specific activity is the activity per unit mass or volume of the radioactive material, typically expressed in $Bq/g\ or\ Bq/mL.$

Applications: Radioactive activity is important in various fields, such as nuclear medicine, radiation detection, and environmental monitoring.

2.4. Half-life:

The relationship between half-life and decay constant is fundamental in understanding and predicting the behavior of radioactive materials, with applications in fields such as nuclear physics, nuclear medicine, and radiometric dating.

The half-life ($T_{1/2}$) is the time it takes for half of the radioactive nuclei to decay: $T_{1/2} = \frac{\ln(2)}{\lambda}$

2.5. Interaction of radiation with matter

When radiation, such as alpha, beta, gamma, or X-rays, interacts with matter, it can undergo various types of interactions, which can lead to the absorption, scattering, or transmission of the radiation. The specific interactions depend on the type of radiation and the properties of the material it is interacting with.

2.5.1. Alpha (α) Radiation:

Alpha particles have a high mass and positive charge, making them strongly ionizing.

They have a short range in matter, typically only a few centimeters in air or a few micrometers in tissue.

Alpha particles can be easily shielded by a thin layer of material, such as a sheet of paper or the outer layer of skin.

2.5.2. Beta (β) Radiation:

Beta particles are high-energy electrons or positrons.

They have a longer range in matter compared to alpha particles, but are less ionizing.

Beta particles can penetrate deeper into materials, but can be shielded by a few millimeters of aluminum or a few centimeters of water.

2.5.3. Gamma (γ) Radiation:

Gamma rays are high-energy electromagnetic radiation, similar to X-rays.

They have a high penetrating power and can travel through significant thicknesses of material.

Gamma radiation can be shielded by dense materials, such as lead or concrete, which absorb the energy of the gamma rays.

2.5.4. X-Rays:

X-rays are also high-energy electromagnetic radiation, similar to gamma rays.

They have a shorter wavelength and higher energy compared to visible light.

X-rays can penetrate matter and are widely used in medical imaging and material analysis.

The interaction of radiation with matter can result in various processes, such as:

Ionization: Radiation can ionize atoms and molecules in the material, creating charged particles and free radicals.

Excitation: Radiation can raise the energy levels of electrons in atoms, causing them to move to higher energy states.

Scattering: Radiation can be scattered by the atoms and molecules in the material, changing the direction of the radiation.

Absorption: Radiation can be absorbed by the material, transferring its energy to the atoms and molecules.

The specific interactions and their effects depend on the type of radiation, the energy of the radiation, and the properties of the material being irradiated. Understanding these interactions is crucial in various fields, such as radiation protection, medical imaging, and materials science. See figure 3, figure 4 and figure 5.

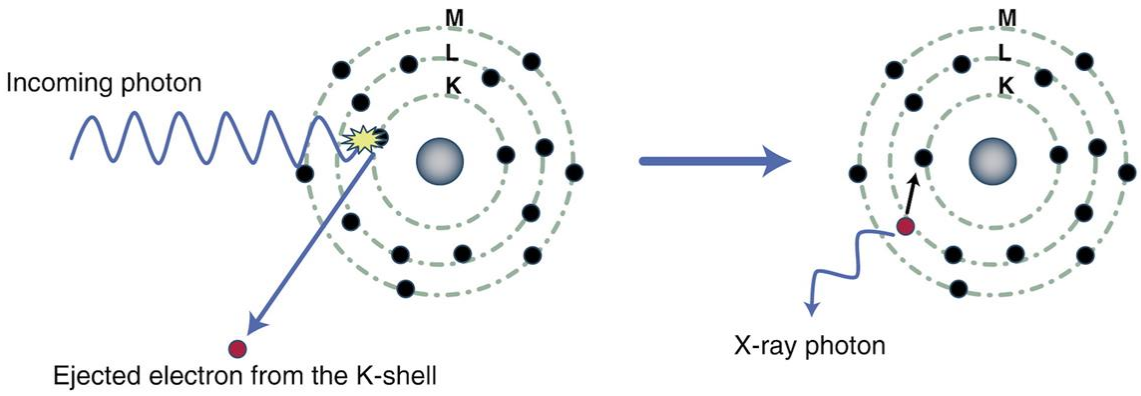


Figure 3: Interaction of Radiation with the Matter Photoelectric effect

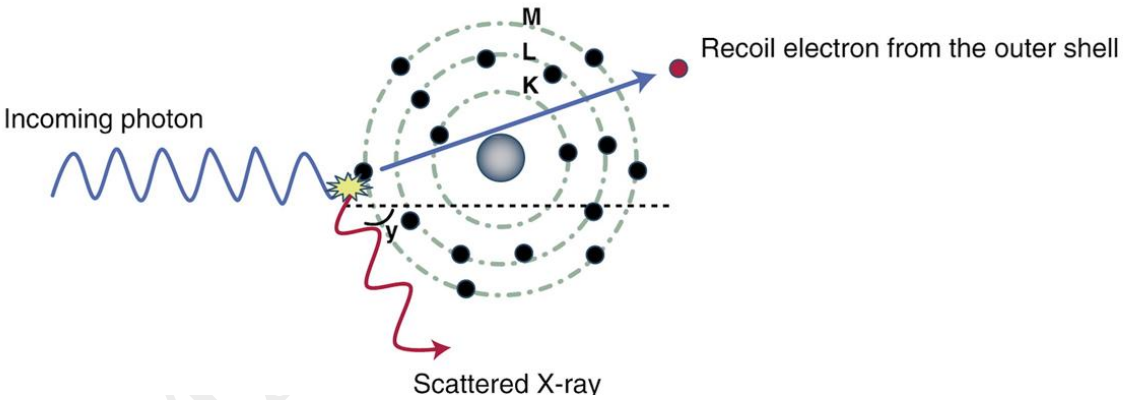


Figure 4: Interaction of Radiation with the Matter Compton effect

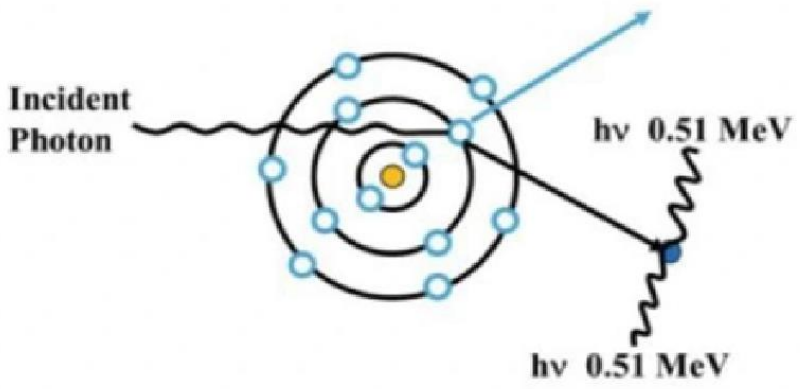


Figure 5: Interaction of Radiation with the Matter pair production

2.5.4.1. Attenuation:

- **Exponential Attenuation:**
  When gamma rays or other types of radiation pass through a medium, their intensity decreases exponentially. This phenomenon is described by the equation:

$$I(x) = I_0 e^{-\mu x}$$

- Where: $I(x)$ is the intensity after traveling a distance x through the medium;
- I_0 is the initial intensity;
- μ is the linear attenuation coefficient, dependent on the material and the energy of the radiation.
- Explanation of terms:
- **Intensity (I):** This refers to the amount of radiation energy passing through a unit area per unit time. It is often measured in units such as counts per minute (CPM) or grays (Gy).
- **Linear attenuation coefficient (μ):** This coefficient represents how easily a material can attenuate radiation. It varies with the type of material and the energy of the radiation. Higher values of μ indicate that the material is more effective at attenuating radiation.

2.5.4.1. Example Calculation:

Let's consider an example where we have a gamma ray source with an initial intensity I_0=1000 CPM, and we want to calculate the intensity after it passes through 5 cm of lead, given that the linear attenuation coefficient for lead (μ) at the gamma ray energy of interest is 10.5cm^{-1}.

Substituting into the Equation:

$$I(5) = 1000CPM \cdot e^{-0.5 \cdot 5} \approx 1000CPM.0.0821 \approx 82.1CPM$$

After passing through 5 cm of lead, the intensity of the gamma rays decreases from 1000 CPM to approximately 82.1 CPM. This example illustrates how effective materials can be in attenuating

radiation and highlights the importance of the linear attenuation coefficient in determining the extent of attenuation.

We can also consider these notes:

Energy dependence: The value of μ is not constant and can change with the energy of the radiation. Higher energy gamma rays may penetrate materials more effectively than lower energy rays.

Material composition: Different materials have different attenuation coefficients. For example, lead is commonly used for shielding against gamma radiation due to its high density and effective attenuation properties.

2.5.5. Photoelectric effect and Compton scattering:

The interaction of radiation (such as gamma rays or X-rays) with matter can occur through various mechanisms, with the photoelectric effect and Compton scattering being two of the most significant. The probability of these interactions is quantified using cross-sections, which are directly related to the linear attenuation coefficient (μ).

Photoelectric Effect

The photoelectric effect occurs when a photon interacts with an atom and is completely absorbed, resulting in the ejection of an electron from the atom. This process is more likely to occur with lower-energy photons and materials with high atomic numbers (Z).

Probability and Cross-Section

The probability of the photoelectric effect occurring can be expressed using the photoelectric cross-section (σ_{PE}), which is a measure of the effective area for interaction between the radiation and the target atoms. The relationship between the photoelectric cross-section and the linear attenuation coefficient is given by:

$$\mu_{PE} = N \cdot \sigma_{PE}$$

Where:

μ_{PE} is the linear attenuation coefficient due to the photoelectric effect.

N is the number density of target atoms (number of atoms per unit volume).

2.5.5.1. Example: Photoelectric Effect

Consider a scenario where a photon with energy of 100 keV interacts with a lead target (Z = 82). The photoelectric cross-section for lead at this energy might be approximately

$\sigma_{PE} \approx 0.1 \text{cm}^2/\text{g}$.

Calculate the Number Density (N): The density of lead is about $11.34 g/cm^3$. The number density N can be calculated as follows:

$$N = \frac{density}{atomic\ mass} \cdot NA$$

Where N_A (Avogadro's number) is approximately 6.022×10^{23} atoms/mol

The atomic mass of lead is approximately $207.2 g/mol$.

$$N \approx 3.25 \times 10^{22} atoms/cm^3$$

$$\mu_{PE} \approx 3.25 \times 10^{21} cm^{-1}$$

Compton Scattering

Compton scattering occurs when a photon collides with a loosely bound or free electron, resulting in a transfer of energy and a change in the direction of the photon. This process is significant for intermediate-energy photons (typically in the range of a few hundred keV).

The probability of Compton scattering is described by the Compton cross-section (σ_C), which also contributes to the linear attenuation coefficient:

$\mu_C = N \cdot \sigma_C$

Where:

μ_C is the linear attenuation coefficient due to Compton scattering.

2.5.5.2. Example: Compton Scattering

Assuming an incident photon energy of 500 keV interacting with the same lead target, the Compton cross-section might be approximately $\sigma_C \approx 0.02 cm^2/g$.

$\mu_C \approx 6.5 \times 10^{20} cm^{-1}$

Both the photoelectric effect and Compton scattering are critical mechanisms for understanding how radiation interacts with matter. The probabilities of these interactions are quantified using cross-sections, which relate directly to the linear attenuation coefficients. Understanding these interactions is essential in fields such as medical imaging, radiation therapy, and radiation safety. See figure 4.

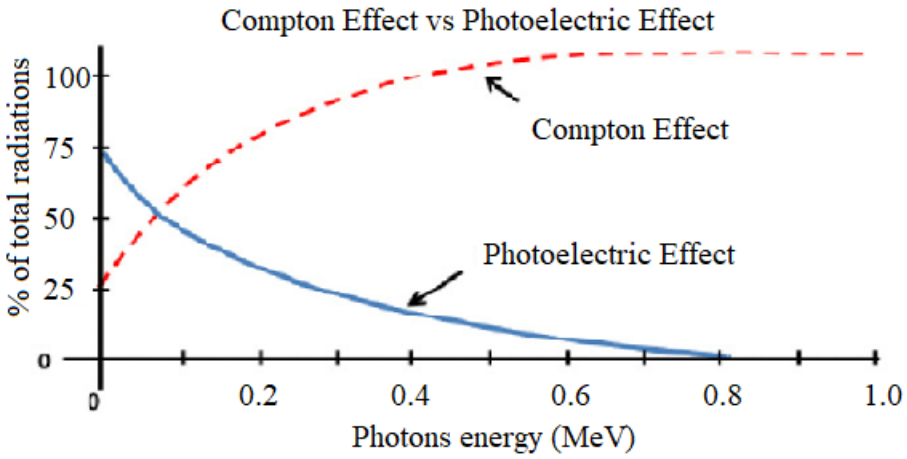


Figure 4: Photoelectric effect and Compton scattering

2.6. Radiopharmaceutical kinetics

Radiopharmaceutical kinetics involves the study of how radiopharmaceuticals distribute, accumulate, and are eliminated in the body. This is crucial for understanding their behavior and optimizing their use in medical imaging and therapy. Compartmental models simplify the complex processes of drug distribution and elimination into manageable mathematical representations.

2.6.1. Compartmental models

The compartmental models provide a framework for understanding the kinetics of radiopharmaceuticals in the body. By using these models, healthcare professionals can better predict the distribution and elimination of these substances, leading to improved diagnostic and therapeutic outcomes.

2.6.1.1. One-compartment model

The simplest model, where the entire body is represented as a single compartment. The substance is assumed to distribute instantaneously throughout the body and is eliminated from this compartment. Useful for substances that distribute quickly and uniformly, such as certain intravenous drugs. The **mathematical representation**:

$$C(t) = C_0 \cdot e^{-k \cdot t}$$

- Where:
- $C(t)$ is the concentration of the substance at time t;
- C_0 is the initial concentration
- k is the elimination rate constant

2.6.1.2 Two-compartment model

The body is divided into two compartments: a central compartment (blood and highly perfused organs) and a peripheral compartment (less perfused tissues). The substance can move between

these compartments and be eliminated from the central compartment. It is commonly used for substances that have a slower distribution phase before reaching equilibrium between compartments. The mathematical representation**:**

$$C(t) = C(t) = A \cdot e^{-\alpha t} + B \cdot e^{-\beta t}$$

- Where:
- A and B are coefficients related to the distribution and elimination phases
- α and β are the rate constants for distribution and elimination phases, respectively

2.6.1.3. Multi-Compartment Model

Extends the two-compartment model by adding more compartments to represent additional tissues or organs. More complex and necessary when the kinetics of the substance cannot be accurately described by simpler models. Applicable in scenarios where the substance has a complex distribution pattern, such as drugs with extensive tissue binding or radiopharmaceuticals targeting multiple tissues.

$$C(t) = \sum_{i=1}^{n} C_i \, e^{-\lambda_i i t}$$

Where:

n: Total number of compartments.

C_i: Coefficient for compartment i.

λ_i: Rate constant for compartment i.

2.6.2. *Applications of compartmental models*

2.6.2.1. Pharmacokinetics:

Drug Distribution: Compartmental models are essential in pharmacokinetics to predict how a drug is distributed throughout the body, its concentration over time, and its eventual elimination. This helps in determining the appropriate dosing regimen.

Therapeutic Monitoring: By modeling the kinetics of a drug, healthcare providers can monitor therapeutic levels in the blood, adjust doses, and avoid toxicity.

2.6.2.2. Nuclear medicine:

Radiopharmaceutical kinetics: Compartmental models help in understanding the behavior of radiopharmaceuticals used in imaging or therapy. For example, in PET imaging, the model predicts the distribution of a radiotracer in the body, helping in interpreting images.

Dosimetry: These models are used to calculate the radiation dose delivered to tissues during radionuclide therapy, ensuring that the target tissue receives the optimal dose while minimizing exposure to healthy tissues.

2.6.2.3. Medical imaging:

- Dynamic imaging: In techniques like dynamic PET (Positron Emission Tomography) or SPECT (Single Photon Emission Computed Tomography), compartmental models help in analyzing the time-activity curves, allowing for the quantification of physiological parameters like blood flow, metabolism, and receptor binding.

2.6.3. Mathematical aspects of compartmental models

2.6.3.2. Differential equations:

Two-Compartment Model is the commonly used model where the radiopharmaceutical is distributed between two compartments (e.g., blood and tissue). The kinetics is described by differential equations:

- $\frac{dC_1(t)}{dt} = -k_{12}C_1(t) + k_{21}C_2(t)$
- $\frac{dC_2(t)}{dt} = k_{12}C_1(t) - k_{21}C_2(t)$

Where: $C_1(t)$ and $C_2(t)$are the concentrations of the radiopharmaceutical in the two compartments

k_{12} and k_{21} are rate constants for the transfer between compartments.

- **Parameter estimation**:

Parameters such as rate constants (k) and initial concentrations are often estimated using experimental data and statistical techniques like least squares fitting. These parameters are crucial for accurately predicting the kinetics of a substance.

- **Simulation and prediction**:

Once a compartmental model is defined and parameters are estimated, simulations can predict the concentration of the substance over time in each compartment. These predictions are used to optimize dosing schedules, imaging protocols, and treatment plans.

2.6.4. Advantages and limitations of compartmental models

2.6.4.1. Advantages:

Simplicity: Provides a simplified framework to understand complex biological processes.

Predictive power: Allows for predictions of how a substance behaves in the body over time, aiding in drug development and medical treatment.

Flexibility: Can be adapted to various substances and scenarios by adjusting the number of compartments and rate constants.

2.6.4.2. Limitations:

Oversimplification: Real biological systems are often more complex than what can be represented by compartmental models. These models may overlook important factors like non-linear kinetics or heterogeneous tissue properties.

Assumptions: The assumptions of homogeneity and first-order kinetics may not always hold true, leading to inaccuracies in predictions.

Parameter estimation: Accurate parameter estimation requires high-quality data, and incorrect estimates can lead to erroneous conclusions.

3. Image reconstruction in nuclear medicine

3.1. Tracer kinetics

Linear model: When using a tracer dose (a small amount of radiopharmaceutical), the kinetics can often be assumed to be linear:

$$C(t) = C_0 e^{-\lambda t}$$

Non-linear models: In cases where higher doses are used or in more complex systems, non-linear models may be necessary.

3.2. SPECT (Single photon emission computed tomography):

Projection data: SPECT involves acquiring projection data at multiple angles. Each projection can be represented mathematically as:

$$P_\theta(x') = \int_{-\infty}^{\infty} \rho(x, y)\, dy$$

- Where: $P_\theta(x')$ is the projection at angle θ along line x'; $\rho(x, y)$ is the radionuclide distribution in the object.

Filtered back projection (FBP): A common method for reconstructing the 2D image from the projections. The reconstructed image $\rho(x, y)$ is given by:

$$\rho(x, y) = \iint_{0-\infty}^{\pi\infty} [P_\theta(x')\,(h\big(x' - x\cos(\theta) - y\sin(\theta)\big)dx']d\theta'$$

- Where: $h(x)$ a filter function.

3.3. PET (Positron emission tomography):

Coincidence detection: PET detects pairs of gamma photons emitted simultaneously in opposite directions. The line of response (LOR) between the detectors is recorded.

Image reconstruction: PET images are reconstructed using algorithms such as filtered back projection or iterative methods like Maximum Likelihood Expectation Maximization (MLEM):

$$\rho(x,y,z) = arg\,\max_{\rho}\{\sum_{i=1}^{N} d_i ln(\rho \cdot R_i) - \rho \cdot R_i\}$$

- Where: d_i is the measured data for detector pair i;
- R_i is the system response function for i;
- $\rho(x,y,z)$ is the reconstructed activity distribution.

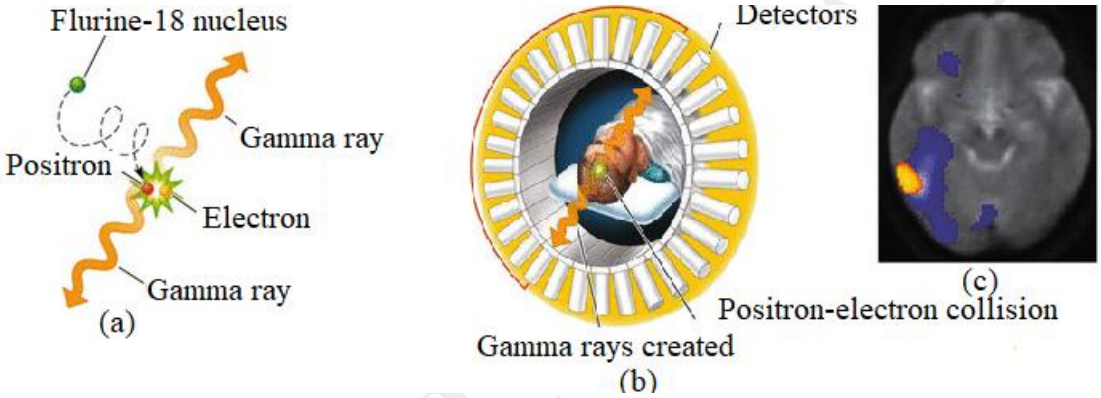


Figure 5: PET imaging reconstruction

Positron emission tomography (PET). (a) Molecules with radioactively labeled probes are injected into the bloodstream. Blood flows preferentially to areas with increased metabolism, such as active brain areas. When the radioactive compound decays, it emits gamma rays in opposite directions. (b) Those gamma rays are detected by sensors surrounding the head of the subject at the PET scanner, providing (c) an indication of the absolute concentration of that compound in different brain regions.

4. Dosimetry

4.1. Absorbed dose

Dose calculation: The absorbed dose D (in grays, Gy) is calculated as:

$$D = A \cdot E \cdot \phi / m$$

- Where: A is the activity administered;
- E is the energy of the emitted radiation;
- ϕ is the fraction of emitted energy absorbed by the tissue;
- m is the mass of the tissue.

4.2. Effective dose

Weighting factors: The effective dose E accounts for the type of radiation and the sensitivity of different tissues:

$$E = \sum_T w_T D_T$$

Where:

w_T is the tissue weighting factor for tissue T

D_T is the absorbed dose to tissue T.

4.3. Application example

4.3.1. Case Study: PET Imaging for cancer diagnosis

A patient is referred for a Positron Emission Tomography (PET) scan to evaluate a suspected malignancy. The radiopharmaceutical used for the PET scan is Fluorodeoxyglucose (FDG), a glucose analog that is preferentially taken up by cancer cells due to their higher metabolic activity.

4.3.2. Dosimetry calculation

The physician decides to administer 10 mCi (millicuries) of FDG.

$$10\,\text{mCi} = 10 \times 37 \times 10^6\,\text{Bq} = 370 \times 10^6\,\text{Bq}$$

The **energy E of emitted radiation (the** average energy of the emitted positrons from FDG) is approximately 0.511 MeV

The **Fraction ϕ of emitted energy absorbed by tissue is about** 60% of the emitted energy is absorbed by the tumor tissue.

The mass of tumor tissue is istimated at 0,5kg

4.3.3. Absorbed dose calcultion

Using the absorbed dose formula:

$$D = A \cdot E \cdot \phi / m$$

$$D \approx 0.51.82 \times 10^{-5} Gy$$

4.3.4. Effective dose calculation

Assuming the patient has multiple tissues exposed to radiation, we consider the absorbed doses to various organs and their respective tissue weighting factors:

Tissue Absorbed Doses:

Tumor: $DT = 3.64 \times 10^{-5}$Gy

Heart: $DT = 1.0 \times 10^{-5}$Gy

Liver: $DT=2.0\times10^{-5}$Gy

Tissue Weighting Factors (wT**):**

Tumor: $w_T=0.12$

Heart: $w_T=0.01$

Liver: $w_T=0.05$

Effective Dose Calculation

Using the effective dose formula:

$E = \sum_T w_T . D_T$

Calculating for each tissue

$E=(0.12\cdot3.64\times10^{-5})+(0.01\cdot1.0\times10^{-5})+(0.05\cdot2.0\times10^{-5})$

Calculating each term

Tumor: $\approx4.37\times10^{-6}$

Heart: $=1.0\times10^{-7}$

Liver: $=1.0\times10^{-6}$

$E\approx5.47\times10^{-6}$Sv (Sv the effective dose unit).

In this example, dosimetry calculations help quantify the absorbed and effective doses of radiation received by the patient during a PET scan. This information is crucial for assessing the risks associated with the procedure, guiding treatment decisions, and ensuring patient safety. By understanding the dosimetry, healthcare professionals can optimize the use of radiopharmaceuticals to achieve the best diagnostic outcomes while minimizing potential harmful effects.

Chapter 5. Optical imaging:

1. Introduction

Optical imaging is a versatile and powerful tool in both clinical and research settings, offering high-resolution, real-time visualization of tissues, cells, and molecular processes. This technology has gained significant traction in the biomedical field due to its ability to provide detailed insights into biological systems without the harmful effects associated with ionizing radiation. While optical imaging has some limitations in terms of tissue penetration, ongoing advancements in technology and techniques are expanding its capabilities, making it an increasingly valuable tool in modern medicine and biomedical research.

Optical imaging employs various light wavelengths to visualize tissues, cells, and molecules within the body. Unlike modalities that rely on ionizing radiation, such as X-rays or CT scans, optical imaging primarily utilizes non-ionizing light in the visible, ultraviolet (UV), or near-infrared (NIR) spectrum. This characteristic allows for safer imaging protocols, particularly in sensitive populations such as pediatric patients and pregnant women. The non-invasive nature of optical imaging not only enhances patient safety but also facilitates repeated imaging sessions, which is crucial for monitoring disease progression or treatment response.

The high-resolution images produced by optical imaging techniques enable researchers and clinicians to observe dynamic biological processes in real time. This capability is particularly beneficial in fields such as oncology, where real-time visualization of tumor responses to therapies can guide treatment decisions. Additionally, optical imaging plays a pivotal role in understanding cellular mechanisms and molecular interactions, thereby contributing to the development of targeted therapies and personalized medicine.

For biomedical engineers, the interest in optical imaging lies in its potential for innovation and improvement in medical diagnostics and therapeutics. As advancements in imaging technologies, such as fluorescence imaging, multispectral imaging, and optical coherence tomography, continue to emerge, biomedical engineers are presented with opportunities to design and optimize imaging systems that enhance diagnostic accuracy and therapeutic efficacy. Furthermore, the integration of optical imaging with other modalities, such as MRI or PET, can provide comprehensive insights into complex biological phenomena, paving the way for more effective treatment strategies.

In summary, optical imaging represents a rapidly evolving field that holds immense promise for biomedical engineering. Its ability to deliver high-resolution, real-time visualization of biological systems without the risks associated with ionizing radiation makes it an indispensable tool in both clinical and research environments. As technology continues to advance, the role of optical imaging in medicine will undoubtedly expand, offering new avenues for exploration and innovation in the pursuit of improved healthcare outcomes. See figure 1.

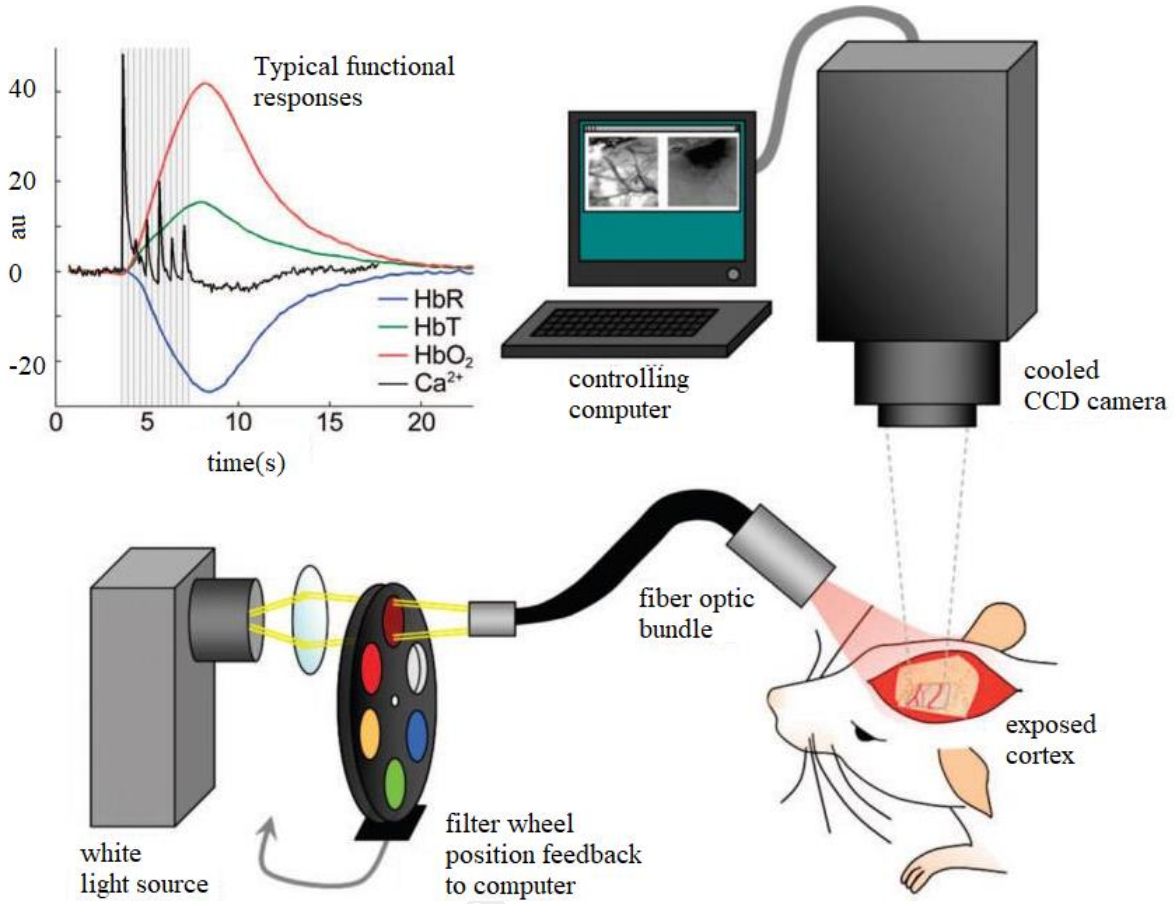


Figure 1: Optical imaging

2. Principles of optical imaging

2.1. Light-tissue interaction

Optical imaging relies on several fundamental interactions between light and biological tissues, each contributing to the overall imaging process. Understanding these interactions is crucial for optimizing imaging techniques and interpreting results effectively. Below are the key principles of optical imaging, along with relevant equations that describe these phenomena.

2.2.1. Absorption

Different tissues and molecules absorb light at specific wavelengths. This absorption can be quantified using the Beer-Lambert Law, which relates the absorbance of light to the properties of the material through which the light is traveling. The equation is given by:

$$A = log10(\frac{I_0}{I}) = \epsilon cl$$

Where:

A = Absorbance (no units)

I_0 = Intensity of incident light

I = Intensity of transmitted light

ϵ = Molar absorptivity (extinction coefficient) of the substance (L/(mol·cm))

c = Concentration of the absorbing species (mol/L)

l = Path length through which the light travels (cm)

This equation indicates that the absorbance increases with the concentration of the absorbing species and the path length, allowing for the identification and quantification of specific molecules or tissue types.

2.1.2. Scattering

As light passes through tissues, it scatters in different directions due to variations in tissue density and composition. The scattering can be described using the Rayleigh scattering equation, particularly for small particles:

$$I \propto \frac{1}{\lambda^4}$$

Where:

I = Intensity of scattered light

λ = Wavelength of the incident light

This relationship shows that shorter wavelengths scatter more than longer wavelengths, which can affect image resolution. Additionally, the scattering coefficient (μ_s) can be defined, which describes how much light is scattered per unit distance:

$$\mu_s = \frac{1}{I_s}$$

Where:

l_s = Mean free path of light in the medium (cm)

2.1.3. Fluorescence

Fluorescence occurs when certain molecules absorb light and re-emit it at a longer wavelength. The fluorescence intensity can be described by the following equation:

$$F = \phi \cdot \epsilon \cdot c \cdot I_0$$

Where:

F = Fluorescence intensity

ϕ = Quantum yield of the fluorescent molecule (no units)

ϵ = Molar absorptivity at the excitation wavelength (L/(mol·cm))

c = Concentration of the fluorescent species (mol/L)

I_0 = Intensity of the excitation light

This equation illustrates that the fluorescence intensity is proportional to the concentration of the fluorescent molecule and the intensity of the incident light, making it a powerful tool for tagging and visualizing specific molecules or structures.

2.1.4. Reflectance

Light that is not absorbed or scattered may be reflected back, and this reflected light can be captured to form an image. The reflectance (R) can be described using the Fresnel equations, which account for the angle of incidence and the refractive indices of the two media involved:

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

Where:

R = Reflectance (no units)

n_1 = Refractive index of the first medium (e.g., air)

n_2 = Refractive index of the second medium (e.g., tissue)

This equation indicates that reflectance depends on the refractive indices of the media, which can vary significantly among different tissue types. The captured reflected light forms the basis of the imaging process, contributing to the overall quality and clarity of the resulting images.

Understanding these principles absorption, scattering, fluorescence, and reflectance along with their corresponding equations, is essential for biomedical engineers and researchers working with optical imaging technologies. These interactions not only influence the quality of the images obtained but also provide critical information about the biological processes occurring within tissues. As technology advances, the ability to manipulate and optimize these interactions will enhance the diagnostic and therapeutic capabilities of optical imaging in clinical and research settings.

2.2. Spectral range

Optical imaging employs various wavelengths of light, each with unique properties and applications. Understanding the interaction of these wavelengths with biological tissues is essential for optimizing imaging techniques. Below are the key types of light used in optical imaging, along with relevant equations that describe their interactions with tissues.

2.2.1. Visible Light

Visible light, typically ranging from 400 to 700 nm, is primarily used for surface imaging, such as skin or eye examinations. The interaction of visible light with tissues can be described by the absorption and scattering coefficients, which influence image quality. The total attenuation of light as it travels through a medium can be represented by the Beer-Lambert Law:

$$I = I_0 e^{-\mu al - \mu sl}$$

Where:

I = Intensity of light after passing through the tissue

I_0 = Initial intensity of light

μ_a = Absorption coefficient (cm^{-1})

μ_s = Scattering coefficient (cm^{-1})

l = Path length through the tissue (cm)

This equation shows how both absorption and scattering reduce the intensity of visible light, impacting the resolution and quality of surface images.

2.2.2. Near-Infrared (NIR)

Near-Infrared (NIR) light, typically ranging from 650 to 900 nm, penetrates deeper into tissues than visible light. This property makes it particularly useful for imaging deeper structures, such as blood vessels and organs. The penetration depth (�d) of light in biological tissues can be estimated using the following equation:

$$d \approx \frac{1}{\mu a + \mu s}$$

Where:

d = Penetration depth (cm)

μ_a = Absorption coefficient (cm^{-1-1})

μ_s = Scattering coefficient (cm^{-1-1})

This equation indicates that as the absorption and scattering coefficients decrease, the penetration depth increases, allowing NIR light to access deeper tissue layers effectively.

2.2.3. Ultraviolet (UV)

Ultraviolet (UV) light, typically ranging from 10 to 400 nm, can induce fluorescence in certain molecules, making it useful for specific imaging applications. However, UV light is generally limited to surface imaging due to its limited penetration depth and potential to cause damage to

biological tissues. The penetration depth of UV light can be estimated similarly to that of visible light:

$$d \approx \frac{1}{\mu a + \mu s}$$

In this case, the absorption coefficient (μa) for UV light is usually higher than for visible and NIR light, resulting in a significantly reduced penetration depth. Additionally, the potential for UV-induced damage can be modeled using the following equation, which describes the relationship between exposure time (t) and the dose (D) absorbed by the tissue:

$$D = I_0 \cdot t \cdot e^{-\mu a l}$$

Where:

D = Dose of UV light absorbed (J/cm²)

I_0 = Intensity of incident UV light (W/cm²)

t = Exposure time (s)

μa = Absorption coefficient (cm^{-1})

l = Path length through the tissue (cm)

This equation highlights the risk of damage from UV light exposure, as higher intensities and longer exposure times can lead to increased doses absorbed by the tissue.

The use of visible light, near-infrared light, and ultraviolet light in optical imaging provides a range of capabilities for visualizing biological structures. Each type of light has distinct properties that influence its application, with equations describing how these wavelengths interact with tissues. Understanding these interactions is crucial for biomedical engineers and researchers aiming to optimize imaging techniques and improve diagnostic and therapeutic outcomes. As technology continues to advance, the ability to harness these different wavelengths will enhance the versatility and effectiveness of optical imaging in clinical and research settings.

2.3. Optical techniques

Optical imaging encompasses various techniques that utilize light to visualize biological structures and processes. Each method has unique principles and applications, often described mathematically. Below are the key optical imaging techniques, along with relevant equations that elucidate their underlying mechanisms.

2,3,1. Reflectance Imaging

Reflectance imaging measures the intensity of light reflected from tissues to create an image. This technique is commonly used in dermatology and ophthalmology. The intensity of the reflected

light (I_r) can be described using the Fresnel equations, which account for the refractive indices of the tissue and the medium (e.g., air):

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

Where:

R = Reflectance (no units)

n_1 = Refractive index of the first medium (e.g., air)

n_2 = Refractive index of the second medium (e.g., tissue)

$$I_r = R \cdot I_0$$

Where:

I_r = Intensity of reflected light

I_0 = Intensity of incident light

This relationship indicates that the amount of light reflected from tissues is dependent on both the incident light intensity and the reflectance coefficient.

2.3.2. Fluorescence Imaging

Fluorescence imaging employs fluorescent dyes or proteins that emit light at specific wavelengths when excited by a light source. The fluorescence intensity (F) can be quantified using the following equation:

$$F = \phi \cdot \epsilon \cdot c \cdot I_0$$

Where:

F = Fluorescence intensity

ϕ = Quantum yield of the fluorescent molecule (no units)

ϵ = Molar absorptivity at the excitation wavelength (L/(mol·cm))

c = Concentration of the fluorescent species (mol/L)

I_0 = Intensity of the excitation light

The equation illustrates how the fluorescence intensity is directly proportional to the concentration of the fluorescent molecules and the intensity of the excitation light, making it an effective method for visualizing specific biological molecules or cells.

2.3.3. Bioluminescence Imaging

Bioluminescence imaging utilizes the emission of light by living organisms, such as the luciferase enzyme found in fireflies, to study biological processes in vivo. The intensity of bioluminescence (L) can be modeled using the following equation:

$$L = k \cdot [Substrate] \cdot [Enzyme]$$

Where:

L = Bioluminescence intensity (photons/s)

k = Rate constant (specific to the reaction

$[Substrate]$ = Concentration of the substrate (e.g., luciferin) (mol/L)

$[Enzyme]$ = Concentration of the enzyme (e.g., luciferase) (mol/L)

This equation indicates that the intensity of bioluminescence is dependent on the concentrations of both the substrate and the enzyme, allowing for real-time monitoring of biological processes in living organisms.

2.3.4. Optical Coherence Tomography (OCT)

Optical coherence tomography (OCT) is a non-invasive imaging technique that captures cross-sectional images of tissues by measuring the echo time delay of reflected light. The basic principle can be described using the following equation for the time delay (t_d) of light reflecting from a tissue layer:

$$t_d = \frac{2d}{c}$$

Where:

t_d = Time delay (s)

d = Depth of the tissue layer (cm)

c = Speed of light in the medium ($\approx 3 \times 10^{10}$ cm/s in tissue)

The depth resolution (Δz) of OCT can be expressed as:

$$\Delta z = \frac{c}{2\Delta f c}$$

Where:

Δz = Depth resolution (cm)

Δf = Bandwidth of the light source (Hz)

This equation indicates that better depth resolution can be achieved with a light source that has a broader bandwidth, allowing OCT to produce high-resolution cross-sectional images, particularly useful in ophthalmology for imaging the retina.

Optical imaging techniques such as reflectance imaging, fluorescence imaging, bioluminescence imaging, and optical coherence tomography each have distinct principles and mathematical foundations. Understanding these methods and their equations is crucial for biomedical engineers and researchers working to optimize imaging technologies and improve diagnostic and therapeutic applications. As advancements in optical imaging continue, these techniques will play an increasingly vital role in clinical and research settings.

3. Applications of optical imaging

3.1. Clinical applications:

Dermatology: Optical imaging is used to evaluate skin conditions, including the detection of skin cancer, by analyzing light reflectance and absorption properties.

Ophthalmology: OCT is commonly used to diagnose and monitor eye diseases such as glaucoma, macular degeneration, and diabetic retinopathy.

Surgery: Fluorescence imaging can be used during surgery to identify cancerous tissues, map blood flow, or visualize nerves, thereby improving surgical precision.

3.2. Research applications:

Molecular imaging: In research, optical imaging is used to study molecular interactions in real-time, allowing scientists to observe processes such as protein-protein interactions, gene expression, and cell signaling.

Preclinical studies: Bioluminescence and fluorescence imaging are extensively used in animal models to study disease progression, monitor therapeutic interventions, and track the behavior of cells or pathogens.

4. Advantages and limitations of optical imaging

4.1. Advantages:

Non-ionizing radiation: Optical imaging uses light, which is non-ionizing and generally safe, making it suitable for repeated use and imaging sensitive areas such as the eyes or developing embryos.

High resolution: Optical imaging provides high spatial resolution, especially for surface imaging, allowing for detailed visualization of tissues and cellular structures.

Real-time imaging: Many optical imaging techniques offer real-time imaging capabilities, which are crucial for guiding surgeries or monitoring dynamic processes in live tissues.

4.2. Limitations:

Limited penetration depth: Optical imaging, especially with visible light, is generally limited to surface or near-surface imaging due to scattering and absorption by tissues.

Scattering: Scattering of light within tissues can degrade image quality and limit the ability to resolve structures deep within the body.

Contrast agents: Some optical imaging techniques require the use of contrast agents (e.g., fluorescent dyes), which may not be universally available or suitable for all patients.

5. Emerging techniques and future directions

5.1. Multiphoton microscopy:

A technique that uses two or more photons of lower energy (typically NIR) to excite a fluorophore, enabling deeper tissue penetration and reduced photodamage, making it ideal for imaging live tissues in research.

5.2. Photoacoustic imaging:

Combines optical imaging with ultrasound to provide high-resolution images with greater tissue penetration. It works by detecting ultrasound waves generated by the absorption of pulsed laser light by tissues, offering insights into both anatomical and functional properties.

5.3. Super-resolution imaging:

Techniques such as Stimulated Emission Depletion (STED) microscopy and Structured Illumination Microscopy (SIM) allow imaging at resolutions beyond the diffraction limit of light, enabling visualization of cellular structures at the nanometer scale.

5.4. Optical molecular imaging:

Advancements in optical imaging are moving towards the ability to visualize specific molecular targets within tissues, enabling personalized medicine approaches where imaging can guide targeted therapies based on individual molecular profile

6. Imaging techniques:

B-mode imaging, or brightness mode imaging, is a fundamental technique used in ultrasound imaging where the brightness of each pixel in the image corresponds to the amplitude of the reflected echo from that specific location in the tissue. This method provides a two-dimensional representation of the internal structures of the body based on the echoes received from ultrasound waves.

6.1. Relationship between Image Intensity and Echo Amplitude

The image intensity $I(x,y)$ at a given pixel position (x,y) is proportional to the magnitude of the echo amplitude $A(x,y)$:

$$I(x,y) \propto |A(x,y)|$$

This relationship indicates that brighter pixels correspond to stronger echoes, which typically originate from denser or more reflective tissues. The proportionality can be expressed more formally as:

$$I(x,y) = k \cdot |A(x,y)|$$

Where:

$I(x,y)$ = Intensity of the pixel at position (x,y)

$A(x,y)$ = Echo amplitude at position (x,y)

k = Proportionality constant that may include factors such as system gain and calibration settings.

6.2. Echo Amplitude and Tissue Properties

The echo amplitude $A(x,y)$ is influenced by several factors, including the properties of the tissue being imaged. The amplitude can be modeled as:

$$A(x,y) = R(x,y)e^{-\mu l}$$

Where:

$R(x,y)$ = Reflection coefficient at position (x,y)

μ = Attenuation coefficient of the tissue (cm-1_{-1})

l = Path length of the ultrasound wave through the tissue (cm)

The reflection coefficient $R(x,y)$ is determined by the differences in acoustic impedance between the tissue interfaces. The acoustic impedance Z of a tissue is given by:

$$Z = \rho \cdot c$$

Where:

ρ = Density of the tissue (kg/m³)

c = Speed of sound in the tissue (m/s)

The reflection coefficient at an interface between two media can be expressed as:

$$R = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2$$

Where:

Z_1 and Z_2 are the acoustic impedances of the two tissues.

6.3. Image Formation Process

The overall process of B-mode imaging involves several steps, which can be summarized as follows:

Ultrasound Wave Emission: A transducer emits ultrasound waves that propagate through the tissue.

Echo Reception: The waves reflect off tissue interfaces and return to the transducer, where they are received as echoes.

Echo Processing: The received echoes are processed to determine their amplitude and time delay, allowing for the calculation of depth.

Image Construction: The intensity of each pixel in the image is determined based on the amplitude of the corresponding echo, creating a grayscale image representation of the internal structures.

6.4. Time Delay and Depth Calculation

The time delay t for the echo to return can be related to the depth d of the tissue layer from which the echo originated:

$$d = \frac{ct}{2}$$

Where:

c = Speed of sound in the tissue (approximately 1540 m/s in soft tissue)

t = Time taken for the echo to return (s)

This relationship indicates that the depth can be calculated based on the time delay of the received echo.

B-mode imaging is a crucial technique in ultrasound diagnostics, providing real-time visualization of internal structures based on the amplitude of reflected echoes. The relationships between image intensity, echo amplitude, tissue properties, and depth calculation form the basis for understanding how B-mode images are generated and interpreted. This technique plays a vital role in various medical applications, including obstetrics, cardiology, and abdominal imaging, facilitating non-invasive examination of internal organs and tissues.

6.5. Resolution:

The resolution of an ultrasound image is determined by the wavelength of the sound waves and the orifice of the transducer. The axial resolution (R_a) is given by:

$$R_a = \frac{c}{2f}$$

Where:

f is the frequency of the ultrasound wave.

c is the speed of sound in the medium.

The lateral resolution R_l is related to the beam width (W) and the depth (z):

$$R_l = \frac{\lambda z}{W}$$

Where: λ is the wavelength of the ultrasound wave.

6.6. Speckle noise:

Speckle noise is a granular pattern that appears in ultrasound images due to the interference of scattered sound waves. It can be described by a statistical model:

$$S(x, y) = \sum_i A_i cos(k_i \cdot x + \phi_i).$$

Where: A_i is the amplitude of the $i - th$ scatterer:

k_i is the wave vector of the $i - th$ scatterer;

ϕ_i is the phase of the $i - th$ scatterer.

7. Advanced techniques

7.1. Harmonic imaging:

Harmonic imaging enhances image quality by using the second harmonic of the transmitted ultrasound frequency. The received signal at the harmonic frequency $2f_0$ improves contrast and resolution:

$$Sharmonic(t) = A_2 \cos(2\omega_0 t + \phi)$$

Where:

A_2 is the amplitude of the second harmonic.

ω_0 is the angular frequency of the fundamental wave.

7.2. Elastography:

Elastography measures tissue stiffness by analyzing the displacement of tissues under mechanical stress. The strain (ϵ) in the tissue is given by:

$$\epsilon = \frac{\Delta L}{L}$$

Where: ΔL is the change in length of the tissue;

L is the original length of the tissue.

Module 237 : Advanced Biomedical Signal and Image Processing (Cours : 26H , TD : 10H , Activités pratiques : 10H)

The Advanced Biomedical Signal and Image Processing module will cover the following three main parts:

Partie 1 : Introduction to Digital Signal and Image Processing

- Signals and Biomedical Signal Processing
- Fourier Analysis and Applications
- Image Filtering, Enhancement, and Restoration
- Edge Detection and Segmentation of Images
- Wavelet Transform
- Other Signal and Image Processing Methods
- Clustering and Classification.

Partie 2 : Processing of Biomedical Signals

- Electric Activities of the Cell
- Electrocardiogram
- Electroencephalogram
- Electromyogram
- Other Biomedical Signals.

Partie 3 : Processing of Biomedical Images

- Principles of Computed Tomography
- X-Ray Imaging and Computed Tomography
- Magnetic Resonance Imaging
- Ultrasound Imaging
- Positron Emission Tomography
- Other Biomedical Imaging Techniques.